MINIMAL SINGULAR METRICS OF A LINE BUNDLE ADMITTING NO ZARISKI-DECOMPOSITION

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ABSTRACT. We give a concrete expression of a minimal singular metric of a big line bundle on a compact Kähler manifold which is the total space of a toric bundle over a complex torus. In this class of manifolds, Nakayama constructed examples which have line bundles admitting no Zariski-decomposition even after any proper modifications. As an application, we discuss the Zariski-closedness of non-nef loci and the openness conjecture of Demailly and Kollár in this class.

CONTENTS

1. INTRODUCTION

We consider the positivity of a big holomorphic line bundle over a compact Kähler complex manifold. Especially, we are interested in the information related to the obstruction to the nef-ness of the line bundle. Our main result is the explicit construction of a minimal singular metric, or a singular hermitian metric of L with minimal singularities, of a big line bundle L when the manifold X is the total spaces of a smooth projective toric bundle over a complex torus.

In this section, we introduce our result when (X, L) is a Nakayama's example ([14] IV §2.6, see Example 6.4 here), which is one of the most important examples when we study the obstruction to the nef-ness of the line bundle, since it admits no Zariski-decomposition even after any proper modifications. Let E_1 be a sufficiently general smooth elliptic curve

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such as $\mathbb{C}/(\mathbb{Z}+(\pi+\sqrt{-1})\mathbb{Z})$, E_2 be a copy of E_1, and z^j be a coordinate of E_j for $j=1,2$. Let us fix an integer $a>1$, points $p_1\in E_1, p_2\in E_2$, and define the three line bundles $L_j (j=0,1,2)$ over $V=E_1\times E_2$ by

$$\begin{aligned} L_0 &= \mathcal{O}(2F_1-4F_2+2\Delta), \\ L_1 &= \mathcal{O}((a-1)F_1+(a-1)F_2+(a+2)\Delta), \end{aligned}$$

and

$$L_2 = \mathcal{O}((a+3)F_1+(a-3)F_2+a\Delta),$$

where F_1 stands for the prime divisor $\{p_1\}\times E_2\subset V$, F_2 stands for the prime divisor $E_1\times\{p_2\}\subset V$, and Δ stands for the prime divisor $\{(x,y)\in E\times E \mid x=y\}$. Then there is a hermitian metric h_j over L_j whose curvature tensor $\sqrt{-1}\Theta_{h_j}\in c_1(L_j)$ is the harmonic form and each h_j can be denoted as $h_j(\xi,\eta)_{(z^1,z^2)}=e^{-2\pi\varphi_j(z^1,z^2)}\xi\overline{\eta}$, where

$$\begin{aligned} \varphi_0(z^1,z^2) &= (z^1,z^2)\begin{pmatrix} 4 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \\ \varphi_1(z^1,z^2) &= (z^1,z^2)\begin{pmatrix} 2a+1 & -(a+2) \\ -(a+2) & 2a+1 \end{pmatrix}\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \\ \varphi_2(z^1,z^2) &= (z^1,z^2)\begin{pmatrix} 2a+3 & -a \\ -a & 2a-3 \end{pmatrix}\begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \end{aligned}$$

on each small open subset U of V with appropriate local trivialization s^j of L_j on U. Let us set the variety X is the total space of a $\mathbb{P}^2$-bundle $\pi\colon\mathbb{P}(L_0\oplus L_1\oplus L_2)\to V$ over V and $L=\mathcal{O}_{\mathbb{P}(L_0\oplus L_1\oplus L_2)}(1)$. Let U be a sufficiently small open set of V. We use the function

$$\begin{aligned} ([x^0;x^1;x^2],z^1,z^2) &\mapsto [x^0s_0(z^1,z^2);x^1s_1(z^1,z^2);x^2s_2(z^1,z^2)] \\ &\in (\mathbb{C}s^0(z^1,z^2)\oplus\mathbb{C}s^1(z^1,z^2)\oplus\mathbb{C}s^2(z^1,z^2))^*/\mathbb{C}^*=\pi^{-1}(z^1,z^2) \end{aligned}$$

as a coordinate on $\pi^{-1}(U)$, where s_j is a dual section of s^j. Using this coordinate, our main result applied to this example can be stated as followings.

Theorem 1.1. *Let (X,L) be the above example, which is introduced by Nakayama* [14] *and admits no Zariski-decomposition even after any proper modifications. There is a minimal singular metric $h_{\min}$ on L whose local weight function ψ is continuous on $X\setminus\mathbb{P}(L_0)$ and is written by*

$$\psi \sim_{\rm sing} \log\max_{(\alpha,\beta)\in H}(|x^1|^{2\alpha}\cdot|x^2|^{2\beta})$$

at each point in $\mathbb{P}(L_0)$ with the local coordinate $(x^1,x^2,z^1,z^2)=([1;x^1,x^2],z^1,z^2)$, where $H=\{(\alpha,\beta)\in\mathbb{R}^2 \mid \alpha,\beta\geq 0,\ a^2(\alpha+\beta)^2=(1-\alpha)^2+(1-\beta)^2\}$.

This expression enables us to compute the multiplier ideal sheaf $\mathcal{J}(h^t_{\min})$ for each positive number t, whose stalk at $x_0\in X$ is defined by

$$\mathcal{J}(h^t_{\min})_{x_0}=\{f\in\mathcal{O}_{X,x} \mid |f|^2e^{-t\varphi_{\min}} \text{ is integrable around } x\},$$

where $\varphi_{\min}$ is the local weight function of $h_{\min}$ around x_0.

Corollary 1.2. *$\mathcal{J}(h^t_{\min})$ is trivial at any point in $X \setminus \mathbb{P}(L_0)$. For a point $x_0 \in \mathbb{P}(L_0)$, the stalk $\mathcal{J}(h_{\min})_{x_0}$ of the multiplier ideal sheaf is the ideal of $\mathcal{O}_{X,x_0}$ which is generated by the system of the polynomials*

$$\{(x^1)^p(x^2)^q \mid (p+1, q+1) \in \mathrm{Int}(S_t) \cap \mathbb{Z}^2\},$$

where we denote by S_t the set $\{(t\alpha, t\beta) \in \mathbb{R}^2 \mid \alpha, \beta \geq 0,\ a^2(\alpha+\beta)^2 \geq (1-\alpha)^2 + (1-\beta)^2\}$ (For the shape of S_t in this case, see Figure 1).

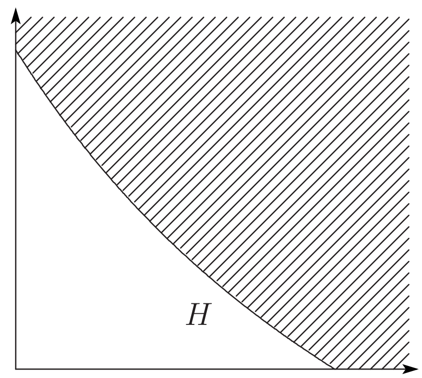

FIGURE 1. The shaded area of this figure represents the set S_1. The set S_t is the set of points $p \in \mathbb{R}^2$ which satisfies $\frac{p}{t} \in S_1$.

According to [14], this (X, L) is an example which admits no Zariski-decomposition even after any proper modifications. So, it can be expected in this case that the behavior of this multiplier ideal sheaf is something different from algebraic cases. Indeed, the set of jumping numbers $\mathrm{Jump}(\psi; x_0)$ for a point x in $\mathbb{P}(L_0)$ (see [9] Section 5 for definition) can be written as following in this case;

$$\mathrm{Jump}(\psi; x_0) = \left\{ \frac{p + \sqrt{2p^2a^2 - q^2}}{2} \,\middle|\, p, q \in \mathbb{Z},\ 0 \leq q < p,\ p - q \equiv 0 \pmod 2 \right\},$$

which is the set of the non-smaller roots of the quadratic equations

$$4T^2 - 4pT + (1 - 2a^2)p^2 + q^2 = 0$$

of T, where integers p and q satisfies the above conditions. This set has different properties from algebraic multiplier ideal sheaf. For example, it seems difficult to expect the "periodicity" property, and does not have the "rationality" property, in this case (For these property, see [9] 1.12 or Remark 6.3 here). Especially, the singularity exponent $c_{x_0}(\psi)$, which is the minimum number in the set of all jumping numbers, satisfies

$$c_{x_0}(\psi) = \sqrt{2}a + 1,$$

and it is clearly irrational.

More generally, we give a concrete expression of a minimal singular metric of a big line bundle L on the total space of such a toric bundle, see Theorem 4.7. As an application, we discuss Zariski-closedness of the non-nef locus $\mathrm{NNef}(L)$ of L, see Corollary 5.5, and the weak openness conjecture of Demailly and Kollár ([7] 5.3 or Conjecture 5.9 here), see Corollary 5.10, in this class.

The organization of the paper is as follows. Let X be the total space of a smooth projective toric bundle over a complex torus, and L be a big line bundle over X. In Section 2, we recall some facts and notations related to analysis on X and L. In Section 3, we fix a way to coordinate X, and study how proper modifications of X or zeros of holomorphic sections of L can be treated by using this coordinate. In Section 4, we construct a singular hermitian metric $\{e^{-\psi_\sigma}\}$ of L and show it is a minimal singular metric. In Section 5, we study some properties related to the positivity of L, as applications of the result in Section 4. Here we introduce how to calculate the Kiselman numbers and the Lelong numbers of minimal singular metrics, and study the non-nef locus of L and multiplier ideal sheaves associated to minimal singular metrics. In Section 6, we introduce three examples for (X, L), all of which is based on the example introduced in [14], and apply our result to them.

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2. Preliminaries to analysis on toric bundles

2.1. **Analysis on compact Kähler manifolds.** In this subsection, let us recall some facts and notations related to analysis on a compact Kähler manifolds X and a holomorhic line bundle L on X.

Let h be a singular hermitian metric of L. Then, for each local trivialization of L on an open set of X, "the inner product" defined by h can be written as

$$(\xi, \eta)_z = e^{-\psi(z)}\xi\overline{\eta}$$

where z is a point in the open set, ξ and η are points in $\mathbb{C}$, which we regard as the z-fiber of L, and ψ is a locally integrable function defined on the open set, which we call the local weight of h.

The local currents written as $dd^c\psi$ for the local weight ψ of h glue together to define the curvature current associated to h. We denote it by $\sqrt{-1}\Theta_h$.

The property "big" and "pseudo-effective" of line bundles are defined in algebraic ways (see [12] 2.2, for example, for details). But, the definitions of these properties can be reworded by using analytic words (see [6] 6.17 for details). Here we recall the reworded definitions.

Definition 2.1. We call L is pseudo-effective if there exists a singular hermitian metric h of L such that $\sqrt{-1}\Theta_h \geq 0$, and call L is big if there exists a singular hermitian metric h of L such that $\sqrt{-1}\Theta_h \geq \varepsilon\omega$ for some positive number ε, where ω is a Kähler form of X.

The next theorem is introduced in [6] as an application of the Ohsawa-Takegoshi extension theorem [15].

Theorem 2.2. ([6] *13.21) Let h be a singular hermitian metric of L satisfying $\sqrt{-1}\Theta_h \geq \varepsilon\omega$ for some positive number ε and a Kähler form ω of X. For sufficiently large integer m, we define the Hilbert space $\mathcal{H}_m$ by*

$$\mathcal{H}_m = H^0(X, L^m \otimes \mathcal{J}(h^m))$$

with the inner product

$$(f, g) = \int_X f\overline{g}e^{-m\varphi}\, dV_\omega,$$

where φ is the local weight of h and dV_ω is the volume form $\frac{1}{m!}\omega^n$, $n = \dim X$. For sufficiently large m, $\mathcal{H}_m$ turns out to be a non zero finite dimensional space.

Moreover, Let us denote an orthonormal basis of $\mathcal{H}_m$ by $(g_{m,k})_{1\leq k\leq N_m}$. Then for every local trivialization of L on an open set U of X, and for every compact subset K of U, there exists positive constants C_1, C_2 independent of m and φ such that

$$\varphi(z) - \frac{C_1}{m} \leq \frac{1}{m}\log\sum_{k=1}^{N_m}|g_{m,k}(z)|^2 \leq \sup_{|x-z|<r}\varphi(z) + \frac{1}{m}\log\frac{C_2}{r^n}$$

holds for all $z \in K$ and $r \leq \frac{1}{2}d(K, X\setminus U)$.

In order to define the minimal singular metric, let us recall how to compare the singularities of plurisubharmonic functions.

Definition 2.3. ([8] 1.4) Let φ and ψ be plurisubharmonic functions defined on a neighborhood of $x \in X$. We write $\varphi \prec_{\mathrm{sing}} \psi$ at x when there exists a constant C such that the inequality $\varphi + C \geq \psi$ holds for each point sufficiently near to x. We denote $\varphi \sim_{\mathrm{sing}} \psi$ at x if $\varphi \prec_{\mathrm{sing}} \psi$ and $\varphi \succ_{\mathrm{sing}} \psi$ holds at x.

By using the notation we defined above, we can define the minimal singular metric as follows.

Definition 2.4. Let $h_{\min}$ be a singular hermitian metric on L which satisfies $\sqrt{-1}\Theta_{h_{\min}} \geq 0$. We call $h_{\min}$ is a minimal singular metric if $\varphi_{\min} \prec_{\mathrm{sing}} \psi$ holds at any point $x \in X$ for all singular hermitian metric h satisfying $\sqrt{-1}\Theta_h \geq 0$, where $\varphi_{\min}$ and ψ stands for the local weight functions of $h_{\min}$ and h, respectively, of a local trivialization of L around the point $x \in X$.

It is known that there exists a minimal singular metric on every pseudo-effective line bundle. This fact is proved by considering the upper envelope of the supremum of the all appropriately normalized ψ's, where ψ is as in Definition 2.4 (see [8] 1.5 for details).

Minimal singular metrics have information related to the positivity of L. In order to explain it, we introduce the definition of the divisorial Zariski-decomposition from [3] when L is big. In this case, the divisorial Zariski-decomposition, which is defined by using the minimal singular metric, coincides with the σ-decomposition, which is defined algebraically in [14].

Let L be a big line bundle. We define the divisor $N(L)$, the negative part of L, by

$$N(L) = \sum_{\Gamma\,:\,\text{prime divisor}} \nu(\varphi_{\min}, \Gamma)\Gamma,$$

where $\varphi_{\min}$ is the local weight of a minimal singular metric of L and $\nu(\varphi_{\min}, \Gamma)$ is the Lelong number of $\varphi_{\min}$ at the divisor Γ. Here, we define the divisorial Zariski-decomposition of L by the decomposition of the cohomology class

$$\{L\} = P(L) + \{N(L)\},$$

where we denote by $\{L\}$ and $\{N(L)\}$ by the cohomology class associated to L and $N(L)$ respectively, and $P(L)$ stands for the cohomology class $\{L\} - \{N(L)\}$, which is called the positive part of L. We call that L admits a Zariski-decomposition if the positive part $P(L)$ is a nef class.

2.2. **Complex tori.** In this paper, we mainly discuss toric bundles over complex tori. So here, let us introduce fundamental terminologies related to complex tori. Let $\Lambda \subset \mathbb{C}^d$ be a lattice. We denote $\mathbb{C}^d/\Lambda$ by V and the natural map $\mathbb{C}^d \to V$ by p.

Proposition 2.5. *([2] Chapter 3) Following four propositions hold for above d, V, and Λ. Here, let us denote by $\mathbb{H}_d$ the set of all hermitian matrices of size $d \times d$ with $\mathbb{C}$-coefficients.*
(1) There exists an injective $\mathbb{R}$-linear map $\mathrm{NS}(V) \otimes \mathbb{R} \to \mathbb{H}_d$.
(2) By this linear map, $\mathrm{NS}(V)$ is identified with the set $\{H \in \mathbb{H}_d \mid \forall\lambda, \mu \in \Lambda, \mathrm{Im}\,(\lambda H \bar{\mu}) \in \mathbb{Z}\}$.
(3) By this linear map, $\mathrm{Nef}(V)$ is identified with the set

$$\{H \in \mathbb{H}_d \mid H \geq 0 \text{ and } H \text{ is an element of the image of the set } \mathrm{NS}(V) \otimes \mathbb{R}\}.$$

(4) Let $c_1(E)$ be identified with $H_E \in \mathbb{H}_d$ by this linear map for a line bundle E on V, and let us denote by h_E the metric on E to which the curvature form associated is a harmonic form with respect to the Euclidean metric. Here we fix a point of V and denote by $z = (z^1, z^2, \dots, z^d)$ a local coordinate of V around the point induced by the map p and the usual coordinate of $\mathbb{C}^d$. Then, there exists an canonically determined local frame e of E on the neighborhood of the point such that, with respect to this local trivialization, the local weight function φ_E of h_E can be written as

$$\varphi_E(z^1, z^2, \dots, z^d) = \pi \cdot (z^1, z^2, \dots, z^d) H_E \overline{\begin{pmatrix} z^1 \\ z^2 \\ \vdots \\ z^d \end{pmatrix}}.$$

2.3. **Toric bundles.** Here, we introduce fundamental terminologies related to toric bundles. We follow [14] IV basically.

Let N be a free $\mathbb{Z}$-module of rank n, and M be the dual module $\mathrm{Hom}(N,\mathbb{Z})$. We denote by $e_1, e_2, \ldots, e_n$ generators of N,and by $e^1, e^2, \ldots, e^n$ the dual generators of M. We write $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ for $N \otimes \mathbb{R}$ and $M \otimes \mathbb{R}$, respectively. We fix a group homomorphism

$$\mathcal{L}\colon M \to \mathrm{Pic}(V)$$

and a fan Σ of N, and construct the toric bundle $\pi\colon \mathbb{T}_N(\Sigma,\mathcal{L}) \to V$. We make an assumption that the fan Σ is smooth projective, which means that the fan is defined by a smooth full-dimensional lattice polytope. Under this assumption, the toric variety $\mathbb{T}_N(\Sigma)$ is a smooth projective variety. Especially, Σ is a smooth complete fan. See [4] Chapter 2 for details. We denote by $\mathcal{L}^m \in \mathrm{Pic}(V)$ the image of $m \in M$. For simplicity, we also denote by $\mathcal{L}^m$ the image of $m \in M_{\mathbb{R}}$ with respect to the linear map

$$\mathcal{L} \otimes \mathbb{R}\colon M_{\mathbb{R}} \to \mathrm{Pic}(V) \otimes \mathbb{R}.$$

Definition 2.6. For $\sigma \in \Sigma$, we define the affine toric bundle $\pi\colon \mathbb{T}_N(\sigma,\mathcal{L}) \to V$ by

$$\mathbb{T}_N(\sigma,\mathcal{L}) = \mathrm{Spec}_V \bigoplus_{m \in \sigma^\vee \cap M} \mathcal{L}^m$$

with the canonical morphism to V, and the toric bundle $\pi\colon \mathbb{T}_N(\Sigma,\mathcal{L}) \to V$ by gluing $\{\mathbb{T}_N(\sigma,\mathcal{L}) \to V\}_{\sigma\in\Sigma}$ in the natural way.

For each cone $\sigma \in \Sigma$, there exists a corresponding $\mathbb{C}^\times$-orbit $\mathbb{O}_\sigma(\mathcal{L})$ as the case of toric varieties. Let us denote by $\mathbb{V}(\sigma,\mathcal{L})$ the closure of $\mathbb{O}_\sigma(\mathcal{L})$ as the subset of $\mathbb{T}_N(\Sigma,\mathcal{L})$. Just as the case of toric varieties, the codimension of $\mathbb{V}(\sigma,\mathcal{L})$ coincides with the dimension of σ. So, especially, for $\sigma \in \Sigma$ which satisfies $\dim \sigma = 1$, $\mathbb{V}(\sigma,\mathcal{L})$ is a prime divisor of $\mathbb{T}_N(\Sigma,\mathcal{L})$.

Definition 2.7. We denote by $\mathrm{Ver}(\Sigma)$ the set of the whole primitive generators $v \in N$ of one-dimensional cones of Σ. For $v \in \mathrm{Ver}(\Sigma)$, we denote by Γ_v the prime divisor $\mathbb{V}(\mathbb{R}_{\geq 0}v,\mathcal{L})$. Let us set

$$\mathrm{SF}_N(\Sigma,\mathbb{Z}) = \{h\colon N_{\mathbb{R}} \to \mathbb{R} \mid \text{for each } \sigma \in \Sigma,\ h|_\sigma \text{ is linear, and } h(N) \subset \mathbb{Z}\}.$$

For $h \in \mathrm{SF}_N(\Sigma,\mathbb{Z})$, we define the divisor D_h by

$$D_h = \sum_{v\in\mathrm{Ver}(\Sigma)} (-h(v))\Gamma_v.$$

As we state in the next proposition, it is known that any line bundle over $\mathbb{T}_N(\Sigma,\mathcal{L})$ can be written by using a divisor with such a form as $\mathcal{O}(D_h)$, and a line bundle which is the pull-back of a line bundle over V.

Proposition 2.8. *([14] 2.3) For any line bundle L on $\mathbb{T}_N(\Sigma,\mathcal{L})$, there exists a line bundle L_0 on V and an element $h \in \mathrm{SF}_N(\Sigma,\mathbb{Z})$ such that*

$$L \cong \pi^* L_0 \otimes \mathcal{O}(D_h),$$

where "$\cong$" stands for the existence of an isomorphism of line bundles.

Example 2.9. The cone $\{0\}$ is always an element of the fan Σ. Let us consider the affine toric bundle $\mathbb{T}_N(\{0\}, \mathcal{L})$. Let U be a sufficiently small open set in V and $z \mapsto s^j(z)$ be such local trivialization of $\mathcal{L}^{e^j}$ on U as in Proposition 2.5, and $z \mapsto s_j(z)$ be the dual frame of the local frame $z \mapsto s^j(z)$ for $j = 1, 2, \dots, n$. It can be easily checked that the frame $z \mapsto s_j(z)$ is also such section of $\mathcal{L}^{-e^j} = (\mathcal{L}^{e^j})^{-1}$ as in Proposition 2.5. Here,

$$\begin{aligned} \mathbb{T}_N(\{0\}, \mathcal{L})|_{\{z\}} &= \operatorname{Spec} \mathbb{C}[s^1(z), s^2(z), \dots, s^n(z), (s^1)^{-1}(z), (s^2)^{-1}(z), \dots, (s^n)^{-1}(z)] \\ &= \prod_{j=1}^{n} \mathbb{C}^\times \cdot s_j(z) \end{aligned}$$

for $z \in U$. Thus, it follows that the affine toric bundle $\mathbb{T}_N(\{0\}, \mathcal{L})$ can be considered as the $(\mathbb{C}^\times)^n$-bundle on V of which the the system $\{s_j\}_j$ works as a local trivialization on U.

Example 2.10. Let us set $n = 2$ and let L_0, L_1, L_2 be line bundles over V. Let $\mathcal{L}$ be a map defined by $e^j \mapsto L_j \otimes L_0^{-1}$ $(j = 1, 2)$ and Σ be the fan generated by the three cones

$$\sigma_1 = \operatorname{Cone}\{e_1, e_2\},\ \sigma_2 = \operatorname{Cone}\{e_2, -(e_1+e_2)\},\ \text{and}\ \sigma_3 = \operatorname{Cone}\{-(e_1+e_2), e_1\}.$$

Let U be a sufficiently small open set in V and $z \mapsto s_1(z), z \mapsto s_2(z)$ be such local trivializations of $(L_1 \otimes L_0^{-1})^{-1}, (L_2 \otimes L_0^{-1})^{-1}$ of U as in Proposition 2.5, respectively, and s^j be the dual of s_j for $j = 1, 2$. Here,

$$\begin{aligned} \mathbb{T}_N(\sigma_1, \mathcal{L})|_{\{z\}} &= \operatorname{Spec} \mathbb{C}[s^1(z), s^2(z)], \\ \mathbb{T}_N(\sigma_2, \mathcal{L})|_{\{z\}} &= \operatorname{Spec} \mathbb{C}[(s^1(z))^{-1}s^2(z), (s^1(z))^{-1}], \\ \mathbb{T}_N(\sigma_3, \mathcal{L})|_{\{z\}} &= \operatorname{Spec} \mathbb{C}[(s^2(z))^{-1}, s^1(z)(s^2(z))^{-1}], \end{aligned}$$

for $z \in U$. Using this expressions, we can calculate that

$$\mathbb{T}_N(\Sigma, \mathcal{L}) = \mathbb{P}(\mathcal{O}_V \oplus (L_1 \otimes L_0^{-1}) \oplus (L_2 \otimes L_0^{-1})) \cong \mathbb{P}(L_0 \oplus L_1 \oplus L_2).$$

In this case, $\operatorname{Ver}(\Sigma)$ is the set consisted by the following three elements;

$$v_0 = -(e_1+e_2),\ v_1 = e_1,\ \text{and}\ v_2 = e_2.$$

Let us define $h \in \mathrm{SF}_N(\Sigma, \mathbb{Z})$ by $v_0 \mapsto -1, v_1 \mapsto 0$, and $v_2 \mapsto 0$. Then the line bundle $L = \mathcal{O}_{\mathbb{P}(L_0 \oplus L_1 \oplus L_2)}(1)$ can be written as

$$L \cong \pi^* L_0 \otimes \mathcal{O}(D_h).$$

$\mathbb{T}_N(\{0\}, \mathcal{L})$, which we considered in Example 2.9, is always a dense subset of $\mathbb{T}_N(\Sigma, \mathcal{L})$. In the case of toric varieties, or the case that V is the "0-dimensional complex torus", regular functions on $\mathbb{T}_N(\{0\}, \mathcal{L})$ can be regarded as meromorphic functions on $\mathbb{T}_N(\{0\}, \mathcal{L})$. There is an analogue of this fact in the general setting.

Definition 2.11. Here, we use notations of Example 2.9. For $m \in M$, we define the meromorphic section χ^m of $\pi^*\mathcal{L}^{-m}$ on $\mathbb{T}_N(\Sigma, \mathcal{L})$ by

$$(x^j \cdot s_j(z))_j \longmapsto \prod_{j=1}^{n} (x^j \cdot s_j(z))^{m_j} = (x^1)^{m_1} \cdot (x^2)^{m_2} \cdot \dots \cdot (x^n)^{m_n} \cdot \left(\prod_{j=1}^{n} (s^j)^{-m_j} \right)(z)$$

on $\mathbb{T}_N(\{0\}, \mathcal{L})|_U$, where $m_j = \langle m, e_j \rangle$.

3. Toric bundles over complex tori

3.1. **Kähler-ness.** First of all, we show that manifolds we consider is Kähler manifolds.

Proposition 3.1. *Let V be a complex torus and Σ be the fan defined by a smooth full-dimensional lattice polytope of M. Then the total space of the toric bundle $\pi\colon \mathbb{T}_N(\Sigma, \mathcal{L}) \to V$ is a Kähler manifolds for all morphism $\mathcal{L}$. Especially, X is a projective manifolds if V is an abelian variety.*

Proof. Let ω be a Kähler form on V. We fix an element $h \in \mathrm{SF}_N(\Sigma, \mathbb{Z})$ such that the corresponding divisor D_h on $\mathbb{T}_N(\Sigma, \mathcal{L})$ is π-ample. The assumption that the fan Σ is defined by a smooth full-dimensional lattice polytope of M yields that there indeed exists such an element h in $\mathrm{SF}_N(\Sigma, \mathbb{Z})$.

Then let $\{e^{-\varphi}\}$ be a smooth hermitian metric on the line bundle $\mathcal{O}(D_h)$ and $\sqrt{-1}\Theta$ be the curvature tensor $dd^c\varphi$ associated to this hermitian metric. Then the form $\omega + \sqrt{-1}\Theta$ defines a Kähler form on $\mathbb{T}_N(\Sigma, \mathcal{L})$, which proves the first half of the statement of the proposition. The latter half of the statement follows by the fact that we can choose ω for the curvature tensor associated to a smooth hermitian metric defined on some ample line bundle over V when V is an abelian variety. $\square$

3.2. **Holomorphic sections and local coordinates.** Here, we also consider holomorphic sections of a line bundle L over the total space of a toric bundle X. By Proposition 2.8, without loss of generality, we may assume

$$L = \pi^* L_0 \otimes \mathcal{O}(D_h),$$

where L_0 is a holomorphic line bundle over V, and h is an element of $\mathrm{SF}_N(\Sigma, \mathbb{Z})$.

Definition 3.2. We define by $\square_h$ the set $\{m \in M_{\mathbb{R}} \mid \forall x \in N_{\mathbb{R}}, \langle m, x \rangle \geq h(x)\}$.

Proposition 3.3. *([4] 4.3.3, [14] 1.16) When V is a 0-dimensional complex torus, the line bundle $\mathcal{O}(D_h)$ is pseudo-effective if and only if the set $\square_h$ is non-empty. In this case,*

$$H^0(X, \mathcal{O}(D_h)) = \bigoplus_{m \in \square_h \cap M} \mathbb{C} \cdot \chi^m.$$

Now, we introduce a proposition which is a relative version of Proposition 3.3.

Definition 3.4. We set

$$\square_{\mathrm{Nef}}(L_0, h) = \{m \in \square_h \mid L_0 \otimes \mathcal{L}^m \text{ is nef}\}$$
$$\square_{\mathrm{PE}}(L_0, h) = \{m \in \square_h \mid L_0 \otimes \mathcal{L}^m \text{ is pseudo-effective}\}$$

for a line bundle L_0 over V and an element $h \in \mathrm{SF}_N(\Sigma, \mathbb{Z})$.

Remark 3.5. For a line bundle L_0 over a complex torus, the condition that L_0 is nef is equivalent to the condition that L_0 is pseudo-effective. So, we obtain the equation $\square_{\mathrm{Nef}}(L_0, h) = \square_{\mathrm{PE}}(L_0, h)$ in our setting.

Since $\square_h$ is a bounded closed convex set, we clearly obtain the following lemma.

Lemma 3.6. *$\square_{\mathrm{Nef}}(L_0, h)$ is a bounded closed convex subset of $M_{\mathbb{R}}$.*

Proposition 3.7. *([14] 2.3, 2.4) The line bundle $L = \pi^* L_0 \otimes \mathcal{O}(D_h)$ is pseudo-effective if and only if the set $\square_{\mathrm{PE}}(L_0, h)$ is non-empty. In this case, we obtain the equation*

$$H^0(X, L) = \bigoplus_{m \in \square_{\mathrm{PE}}(L_0,h) \cap M} \chi^m \cdot \pi^* H^0(V, L_0 \otimes \mathcal{L}^m).$$

Observation 3.8. Let us consider the meromorphic function $\chi^m \cdot \pi^* f$ which appeared in Proposition 3.7 by using notations of Example 2.9. Let U be a sufficiently small open set in V and $z \mapsto s^0(z)$ be such a local trivialization of L_0 on U as in Proposition 2.5. Under the local trivialization $z \mapsto \left(s^0 \cdot \prod_{j=1}^n s^j\right)(z)$ of $L_0 \otimes \mathcal{L}^m$, we may assume f is written as

$$f|_U(z) = \eta(z) \cdot \left(s^0 \cdot \prod_{j=1}^{n} (s^j)^{\langle m, e_j \rangle}\right)(z)$$

on U for some holomorphic function η on U. Since

$$\chi^m \cdot \pi^* f((x^j \cdot s_j(z))_j) = \chi^m((x^j \cdot s_j(z))_j) \cdot f(z) = \left(\prod_{j=1}^{n} (x^j)^{\langle m, e_j \rangle}\right) \eta(z) \cdot s^0(z)$$

holds, it can be checked that $\chi^m \cdot \pi^* f$ is a meromorphic section of $\pi^* L_0$, indeed. Moreover we can check that it is an element of $H^0(X, L) = H^0(X, \pi^* L_0 \otimes \mathcal{O}(D_h))$, since m is an element of $\square_h$.

In Observation 3.8, we calculated $\chi^m \cdot \pi^* f$ as a meromorphic section of $\pi^* L_0$. From now on, we rewrite it as a holomorphic section of $\pi^* L_0 \otimes \mathcal{O}(D_h)$ by using a local coordinate we define next.

Definition 3.9. Let σ be an element of $\Sigma_{\max}$, where we denote by $\Sigma_{\max}$ the set $\{\sigma \in \Sigma \mid \dim \sigma = n\}$ of the whole maximal cones in Σ. Since the fan Σ is smooth, there exists $v_1, v_2, \ldots, v_n \in \mathrm{Ver}(\Sigma)$ such that

$$\sigma = \mathrm{Cone}\{v_1, v_2, \ldots, v_n\}$$

and $v_1, v_2, \ldots, v_n$ generates N. We call such $v_1, v_2, \ldots, v_n$ N-minimal generators of σ.

Let $v^1, v^2, \ldots, v^n$ be the dual generators of $v_1, v_2, \ldots, v_n$. Then the dual cone of σ can be written as $\sigma^\vee = \mathrm{Cone}\{v^1, v^2, \ldots, v^n\}$. Let U be a sufficiently small open set in V. Let us fix such a local trivializations $z \mapsto t^j(z)$ of $\mathcal{L}^{v^j}$ on U as in Proposition 2.5, and the dual section t_j of t^j for $j = 1, 2, \ldots, n$. Using these notations, we can calculate

$$\mathbb{T}_N(\sigma, \mathcal{L})|_{\{z\}} = \mathrm{Spec} \left.\bigoplus_{a_1, a_2, \ldots, a_n \geq 0} \mathcal{L}^{\sum_j a_j v^j}\right|_{\{z\}} = \mathrm{Spec}\, \mathbb{C}[t^1(z), t^2(z), \ldots, t^n(z)]$$

for $z \in U$. So, it turns out that $\mathbb{T}_N(\sigma, \mathcal{L})$ is a $\mathbb{C}^n$-bundle which $t_1, t_2, \ldots, t_n$ gives a local trivialization on U. So, we can regard the map

$$(x^1, x^2, \ldots, x^n, z) \longmapsto (x^j \cdot t_j(z))_j \in \mathbb{T}_N(\sigma, \mathcal{L})|_{\{z\}}$$

as a local coordinate on $\mathbb{T}_N(\sigma, \mathcal{L})|_U$. We call this local coordinate the canonical coordinate of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to the N-minimal generator $v_1, v_2, \ldots, v_n$ of σ.

Remark 3.10. Let $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $(x^1, x^2, \ldots, x^n, z)$ be the canonical coordinate of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \ldots, v_n$. Then,

$$\{x^j = 0\} = \Gamma_{v_j}$$

holds for $j = 1, 2, \ldots, n$ on $\mathbb{T}_N(\sigma, \mathcal{L})|_U$.

Definition 3.11. For $\sigma \in \Sigma_{\max}$, we denote by $m_\sigma \in M$ the point which satisfies $h(w) = \langle m_\sigma, w\rangle$ for all $w \in \sigma$. We call $\{m_\sigma\}_\sigma$ the Cartier data of D_h.

Observation 3.12. Let σ be an element of $\Sigma_{\max}$, $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $(x^1, x^2, \ldots, x^n, z)$ be the canonical coordinate of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \ldots, v_n$. In $\mathbb{T}_N(\sigma, \mathcal{L})|_U$, the map

$$(x^1, x^2, \ldots, x^n, z) \mapsto \prod_{j=1}^n (x^j)^{\langle m_\sigma, v_j\rangle}$$

gives a local trivialization of $\mathcal{O}(D_h)$, where $\{m_\sigma\}_\sigma$ is the Cartier data of D_h. So, by using notations in Observation 3.8,

$$(x^1, x^2, \ldots, x^n, z) \longmapsto \left(\prod_{j=1}^n (x^j)^{\langle m_\sigma, v_j\rangle}\right) \cdot s^0(z)$$

gives a local trivialization of L. Under this trivialization, $\chi^m \cdot \pi^* f \in H^0(X, L)$ can be regarded as the holomorphic function

$$(x^1, x^2, \ldots, x^n, z) \longmapsto \left(\prod_{j=1}^n (x^j)^{\langle m - m_\sigma, v_j\rangle}\right) \cdot \eta(z)$$

on $\mathbb{T}_N(\sigma, \mathcal{L})|_U$.

The projective line $\mathbb{P}^1 = \{[z; w]\}$ can be regarded as the union of two disks $\{[z; 1] \mid |z| \leq 1\}$ and $\{[1; w] \mid |w| \leq 1\}$ with radius 1. The following proposition is an analogy of this fact.

Proposition 3.13. *Let U be a sufficiently small open set in V, z_0 be a point in U, σ be an element of $\Sigma_{\max}$, $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $(x^1, x^2, \ldots, x^n, z)$ be the canonical coordinate of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \ldots, v_n$. We set*

$$K_{\sigma, z_0} = \{(x^1, x^2, \ldots, x^n, z_0) \in \mathbb{T}_N(\sigma, \mathcal{L}) \mid \forall j \in \{1, 2, \ldots, n\}, |x^j| \leq 1\}.$$

Then,

$$\bigcup_{\sigma\in\Sigma_{\max}} K_{\sigma,z_0} = \pi^{-1}(z_0)$$

holds.

Proof. Since $\overline{\mathbb{T}_N(\{0\},\mathcal{L})|_{\{z_0\}}} = \pi^{-1}(z_0)$, it is sufficient to show that

$$\bigcup_{\sigma\in\Sigma_{\max}} K_{\sigma,z_0} \supset \mathbb{T}_N(\{0\},\mathcal{L})|_{\{z_0\}}.$$

Let us fix a point $y_0 \in \mathbb{T}_N(\{0\},\mathcal{L})|_{\{z_0\}}$ and an element $\tau \in \Sigma_{\max}$. Let $u_1, u_2, \dots, u_n$ be N-minimal generators of τ, and $(y^1, y^2, \dots, y^n, z)$ be the canonical coordinate of $\mathbb{T}_N(\tau,\mathcal{L})|_U$ associated to $u_1, u_2, \dots, u_n$. In this coordinate, assume y_0 is written as $(y_0^1, y_0^2, \dots, y_0^n, z_0)$. Since $y_0 \in \mathbb{T}_N(\{0\},\mathcal{L})$, it turns out that $y_0^j \neq 0$ for all j. Thus,

$$w_0 = -\sum_{j=1}^{n} \log|y_0^j| \cdot u_j$$

defines a point of $N_{\mathbb{R}}$. Since Σ is complete, there exists a $\sigma \in \Sigma_{\max}$ such that $n_0 \in \sigma$. Let $v_1, v_2, \dots, v_n$ be N-minimal generators of σ, and $(x^1, x^2, \dots, x^n, z)$ be the canonical coordinate of $\mathbb{T}_N(\sigma,\mathcal{L})|_U$ associated to $v_1, v_2, \dots, v_n$. In this coordinate, y_0 can be written as

$$y_0 = \left(\left(\prod_{k=1}^{n} (y_0^k)^{\langle v^j, u_k\rangle} \right)_j, z_0 \right),$$

where $v^1, v^2, \dots, v^n$ is the dual basis of $v_1, v_2, \dots, v_n$. On the other hands, w_0 can be rewritten as

$$w_0 = -\sum_{k=1}^{n} \log|y_0^k| \cdot u_k = -\sum_{k=1}^{n}\sum_{j=1}^{n} \log|y_0^k| \langle v^j, u_k\rangle \cdot v_j = -\sum_{j=1}^{n} \log\left| \prod_{k=1}^{n} (y_0^k)^{\langle v^j, u_k\rangle} \right| \cdot v_j.$$

Since we have chosen σ as the condition $n_0 \in \sigma$ holds, $-\log|\prod_{k=1}^{n}(y_0^k)^{\langle v^j,u_k\rangle}| \geq 0$, or $|\prod_{k=1}^{n}(y_0^k)^{\langle v^j,u_k\rangle}| \leq 1$ holds for all $j \in \{1, 2, \dots, n\}$. We thus get $y_0 \in K_{\sigma,z_0}$, which proves the proposition. $\square$

3.3. **Modifications.** Let Σ be a smooth projective fan of the n-dimensional lattice N. Here, we fix a smooth subdivision fan $\tilde{\Sigma}$ of Σ, and consider the toric bundle $\tilde{X} = \mathbb{T}_N(\tilde{\Sigma},\mathcal{L})$ and the canonical morphism $\mu\colon \tilde{X} \to X$. As the case of toric varieties, $\mu\colon \tilde{X} \to X$ is a proper modification of X. From this section, we use letters with subscripts such as $v_1, v_2, \dots, v_n$ for generators of N, and we denote the dual generators by the same letters with superscripts, such as $v^1, v^2, \dots, v^n$, throughout this paper.

First of all, we obtain the following result by monotone computations.

Lemma 3.14. *Let $\sigma \in \Sigma_{\max}, \tilde{\sigma} \in \tilde{\Sigma}_{\max}$ be cones such that $\tilde{\sigma} \subset \sigma$, $v_1, v_2, \ldots, v_n$ be N-minimal generators of σ, and $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n$ be N-minimal generators of $\tilde{\sigma}$. We denote by $(x^1, x^2, \ldots, x^n, z)$ and $(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n, z)$ the canonical coordinates of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ and $\mathbb{T}_N(\tilde{\sigma}, \mathcal{L})|_U$, respectively. In these coordinates, the morphism $\mu\colon \tilde{X} \to X$ can be written as*

$$\mu(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n, z) = \left(\left(\prod_{k=1}^{n} (\tilde{x}^k)^{\langle v^j, \tilde{v}_k \rangle} \right)_j, \ z \right).$$

Lemma 3.14 yields the next corollary.

Corollary 3.15. *Here, we use notations in Lemma 3.14. For $j \in \{1, 2, \ldots, n\}$, there exists a subset $J_{v_j} \subset \{1, 2, \ldots, n\}$ such that*

$$\mu^* \Gamma_{v_j} = \bigcup_{k \in J_{v_j}} \{\tilde{x}^k = 0\}$$

in $\mathbb{T}_N(\tilde{\sigma}, \mathcal{L})|_U$.

Remark 3.16. For Corollary 3.15, the set J_{v_j} can be written as

$$J_{v_j} = \{k \in \{1, 2, \ldots, n\} \mid \langle v^j, \tilde{v}_k \rangle \neq 0\}.$$

For $\sigma \in \Sigma_{\max}$, we define the set $\tilde{\Sigma}_\sigma$ by

$$\tilde{\Sigma}_\sigma = \{\tilde{\sigma} \in \tilde{\Sigma} \mid \tilde{\sigma} \subset \sigma\},$$

and we denote by $(\tilde{\Sigma}_\sigma)_{\max}$ the set $\{\tilde{\sigma} \in \tilde{\Sigma}_\sigma \mid \dim \tilde{\sigma} = n\}$. By using the expression of μ in Lemma 3.14, we can get the following lemma.

Lemma 3.17. *Let us fix a point $z_0 \in U$, a set $I \subset \{1, 2, \ldots, n\}$, and a cone $\sigma \in \Sigma_{\max}$. We set*

$$W_{I,\sigma,z_0} = \{(x^1, x^2, \ldots, x^n, z_0) \in \mathbb{T}_N(\sigma, \mathcal{L}) \mid \forall j \in I, |x^j| \leq 1, \ \forall j \in \{1, 2, \ldots, n\}, x^j \neq 0\},$$

and denote by $W_{I,\tilde{\sigma},z_0}$ the set

$$\{(\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n, z_0) \in \mathbb{T}_N(\tilde{\sigma}, \mathcal{L}) \mid \forall k \in \cup_{j \in I} J_{v_j}, |\tilde{x}^k| \leq 1, \ \forall j \in \{1, 2, \ldots, n\}, \tilde{x}^j \neq 0\}$$

for each $\tilde{\sigma} \in (\tilde{\Sigma}_\sigma)_{\max}$. Here, we are using notations in Lemma 3.14 and Corollary 3.15. Then,

$$\mu\left(\bigcup_{\tilde{\sigma} \in (\tilde{\Sigma}_\sigma)_{\max}} W_{I,\tilde{\sigma},z_0} \right) = W_{I,\sigma,z_0}$$

holds.

This lemma can be proved in the almost same way as those used in Lemma 3.13. Applying this lemma with $I = \{1, 2, \ldots, n\}$, we obtain the next corollary.

Corollary 3.18. *Here we use notations in Lemma 3.17. Let us set*

$$K_\sigma = \{(x^1, x^2, \dots, x^n, z) \in \mathbb{T}_N(\sigma, \mathcal{L})|_{\overline{U}} \mid \forall j \in \{1, 2, \dots, n\}, |x^j| \le 1\}$$

and

$$K_{\tilde\sigma} = \{(\tilde x^1, \tilde x^2, \dots, \tilde x^n, z) \in \mathbb{T}_N(\tilde\sigma, \mathcal{L})|_{\overline{U}} \mid \forall j \in \{1, 2, \dots, n\}, |\tilde x^j| \le 1\}$$

for each n-dimensional cone $\tilde\sigma \in \tilde\Sigma_\sigma$. Then,

$$\mu\left(\bigcup_{\tilde\sigma \in (\tilde\Sigma_\sigma)_{\max}} K_{\tilde\sigma}\right) = K_\sigma$$

holds.

3.4. **Convex subsets of M.** Let Σ be a smooth projective fan of the n-dimensional lattice N, $\sigma \in \Sigma$ be a n-dimensional cone, $v_1, v_2, \dots, v_n$ be a N-minimal generators of σ, and $(x^1, x^2, \dots, x^n, z)$ be the canonical coordinate of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \dots, v_n$. Here, U is a sufficiently small open set in V.

Definition 3.19. For $A \subset \sigma^\vee$, we define the set $\overline{\overline{A}}$ by

$$\overline{\overline{A}} = \{m \in \sigma^\vee \mid \forall w \in \sigma, \min_{m' \in A} \langle m', w\rangle \le \langle m, w\rangle\}.$$

When $A = \emptyset$, we regards $\overline{\overline{\emptyset}}$ as $\sigma^\vee$, formally.

Definition 3.20. Let m_σ be an element of the Cartier data D_h which is associated to σ. We define the set $S(L_0, h) \subset \sigma^\vee$ by

$$S(L_0, h)_\sigma = \overline{\overline{\{m - m_\sigma \mid m \in \square_{\mathrm{Nef}}(L_0, h)\}}}.$$

Remark 3.21. In $\prod_{j\in I}\{|x^j| < 1\} \times \prod_{j \notin I}\{x^j \in \mathbb{C}\} \times U$,

$$\max_{m \in S(L_0,h)_\sigma} \prod_{j\in I} |x^j|^{2\langle m, v_j\rangle} = \max_{m \in \square_{\mathrm{Nef}}(L_0,h)} \prod_{j \in I} |x^j|^{2\langle m - m_\sigma, v_j\rangle}$$

for any $I \subset \{1, 2, \dots, n\}$, where m_σ is an element of the Cartier data D_h which is associated to σ. $\square$

Definition 3.22. For a point $(x_0^1, x_0^2, \dots, x_0^n, z_0) \in \mathbb{T}_N(\sigma, \mathcal{L})|_U$, let us define the set I by $I = \{j \in \{1, 2, \dots, n\} \mid x_0^j = 0\}$. Here, we define the set $P(f_1, f_2, \dots, f_l)_{(x_0^1, x_0^2, \dots, x_0^n, z_0)}$ for $f_1, f_2, \dots, f_l \in \mathcal{O}_{(x_0^1, x_0^2, \dots, x_0^n, z_0)}$ as follows.

Let the Taylor expansion of each f_ν $(\nu = 1, 2, \dots, l)$ around the point $(x_0^1, x_0^2, \dots, x_0^n, z_0)$ for variables $\{x^j\}_{j \in I}$ be

$$f_\nu(x^1, x^2, \dots, x^n) = \sum_{\alpha \ge 0} (x^I)^\alpha A_{\nu,\alpha}(x^{I^c}, z),$$

where $\alpha = (a_j)_{j\in I}$ is a multi-index, the signature "$(x^I)^\alpha$" stands for $\prod_{j\in I}(x^j)^{a_j}$, and $A_{\nu,\alpha}$ is the germ of a holomorphic function with $(n - \#I + d)$-variables $(x^{I^c}, z) = ((x^j)_{j\notin I}, z)$. Using these letters, we define $P(f_1, f_2, \ldots, f_l)_{(x_0^1,x_0^2,\ldots,x_0^n,z_0)}$ by

$$P(f_1, f_2, \ldots, f_l)_{(x_0^1,x_0^2,\ldots,x_0^n,z_0)} = \overline{\overline{\bigcup_{\nu=1}^{l}\left\{\sum_{j\in I} a_j\cdot v^j \,\middle|\, A_{\nu,(a_j)_j}\not\equiv 0\right\}}} \subset \sigma^\vee.$$

Remark 3.23. Here, we use notations in Definition 3.22. Set

$$P_\sigma = P(f_1, f_2, \ldots, f_l)_{(0,0,\ldots,0,z_0)}$$

for $(0, 0, \ldots, 0, z_0) \in \mathbb{T}_N(\sigma, \mathcal{L})|_U$. Let $\tilde\Sigma$ be a smooth complete fan which is a subdivision of Σ, $\tilde\sigma \in \tilde\Sigma_{\max}$ be a cone such that $\tilde\sigma \subset \sigma$, $\tilde v_1, \tilde v_2, \ldots, \tilde v_n$ be N-minimal generators of $\tilde\sigma$, and $(\tilde x^1, \tilde x^2, \ldots, \tilde x^n, z)$ be the canonical coordinate of $\mathbb{T}_N(\tilde\sigma, \mathcal{L})|_U$ associated to $\tilde v_1, \tilde v_2, \ldots, \tilde v_n$. For the point $(0, 0, \ldots, 0, z_0)$, let us set

$$P_{\tilde\sigma} = P(\mu^*f_1, \mu^*f_2, \ldots, \mu^*f_l)_{(0,0,\ldots,0,z_0)},$$

and assume that f_ν is expanded as

$$f_\nu(\tilde x^1, \tilde x^2, \ldots, \tilde x^n, z) = \sum_{(a_j)_j\geq 0}\prod_{j=1}^{n}(x^j)^{a_j}A_{\nu,(a_j)_j}(z)$$

around $(0, 0, \ldots, 0, z_0)$. Then, by Lemma 3.14, μ^*f_ν can be written as

$$\mu^*f_\nu(\tilde x^1, \tilde x^2, \ldots, \tilde x^n, z) = \sum_{(a_j)_j\geq 0}\prod_{k=1}^{n}(\tilde x^k)^{\sum_{j=1}^n a_j\langle v^j,\tilde v_k\rangle}A_{\nu,(a_j)_j}(z)$$

around $(0, 0, \ldots, 0, z_0)$. Thus, it follows that the following two sets are same;

$$\bigcup_{\nu=1}^{l}\left\{\sum_{j=1}^{n} a_j\cdot v^j \,\middle|\, A_{\nu,(a_j)_j}\not\equiv 0\right\} = \bigcup_{\nu=1}^{l}\left\{\sum_{j,k=1}^{n} a_j\langle v^j, \tilde v_k\rangle\cdot \tilde v^k \,\middle|\, A_{\nu,(a_j)_j}\not\equiv 0\right\}.$$

But, since the two signature $\overline{\overline{\cdot}}$ appeared in the definition of P_σ and $P_{\tilde\sigma}$ are different from each other, we can not obtain nothing more than the inclusion $P_\sigma \subset P_{\tilde\sigma}$ about these sets in general.

Remark 3.24. Here, we use notations appeared in Definition 3.22.

We remark that $P(f_1, f_2, \ldots, f_l)_{(x_0^1,x_0^2,\ldots,x_0^n,z_0)}$ is finitely generated in the following sense; There exists finite subset

$$\{m_1, m_2, \ldots, m_l\} \subset P(f_1, f_2, \ldots, f_l)_{(x_0^1,x_0^2,\ldots,x_0^n,z_0)} \cap \bigoplus_{j=1}^{n}\mathbb{Z}_{\geq 0}v^j$$

of the lattice such that the equation

$$P(f_1, f_2, \ldots, f_l)_{(x_0^1,x_0^2,\ldots,x_0^n,z_0)} = \overline{\overline{\{m_1, m_2, \ldots, m_l\}}}$$

holds. More generally, for any subset

$$A \subset \bigoplus_{j=1}^{n} \mathbb{Z}_{\geq 0} v^j,$$

there exists finite subset $\{m_1, m_2, \dots, m_l\} \subset \overline{\overline{A}} \cap \bigoplus_{j=1}^{n} \mathbb{Z}_{\geq 0} v^j$ of lattice points such that the equation

$$\overline{\overline{A}} = \overline{\overline{\{m_1, m_2, \dots, m_l\}}}$$

holds.

Lemma 3.25. *For any finite set $A \subset \bigoplus_{j=1}^{n} \mathbb{Q}_{\geq 0} v^j$ of rational points, there exists a smooth complete cone $\tilde{\Sigma}$ which satisfies the following conditions; $\tilde{\Sigma}$ is a subdivision of Σ, and for all n-dimensional cone $\tilde{\sigma} \in \tilde{\Sigma}$ satisfying $\tilde{\sigma} \subset \sigma$, there exists an element $m_0 \in A$ such that*

$$\min_{m \in \overline{\overline{A}}} \langle m, w \rangle = \langle m_0, w \rangle$$

holds for all $w \in \tilde{\sigma}$, where $\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n$ is N-minimal generators of $\tilde{\sigma}$.

Proof. Let $\tilde{\Sigma}$ be a fan which is made by cutting all cones of Σ by the all hyperplanes

$$\{w \in N_{\mathbb{R}} \mid \langle m_j, w \rangle = \langle m_k, w \rangle\} \ (m_j, m_k \in A)$$

of $N_{\mathbb{R}}$. Since $A \subset \bigoplus_{j=1}^{n} \mathbb{Q}_{\geq 0} v^j$, each cone of $\tilde{\Sigma}$ is rational. Moreover, for all n-dimensional cone of $\tilde{\Sigma}$ satisfying $\tilde{\sigma} \subset \sigma$, there exists an element $m_{\tilde{\sigma}} \in A$ such that

$$\forall w \in \tilde{\sigma}, \ \min_{m \in \overline{\overline{A}}} \langle m, w \rangle = \langle m_{\tilde{\sigma}}, w \rangle.$$

Let $\tilde{\Sigma}'$ be a smooth fan which is a subdivision of $\tilde{\Sigma}$. This fan $\tilde{\Sigma}'$ is what we desired. $\square$

4. Construction of minimal singular metrics

Here, we use notations in the previous section. In this section, we construct a minimal singular metric of the big line bundle $L = \pi^* L_0 \otimes \mathcal{O}(D_h)$ over the total space of a toric bundle $X = \mathbb{T}_N(\Sigma, \mathcal{L})$ over a complex torus V, where Σ is a smooth projective fan in a n-dimensional fan N. According to Proposition 3.7, it is clear that the set $\square_{\mathrm{PE}}(L_0, h) = \square_{\mathrm{Nef}}(L_0, h)$ is not empty in this setting.

First of all, we define the singular hermitian metric $e^{-\psi_{\sigma,m}}$ for each $m \in \square_{\mathrm{Nef}}(L_0, h)$.

Definition 4.1. Let m be an element of $\square_{\mathrm{Nef}}(L_0, h)$, σ be an element of $\Sigma_{\max}$, $v_1, v_2, \dots, v_n$ be N-minimal generators of σ, and $\{m_\sigma\}_\sigma$ be the Cartier data of D_h. Here, we define the plurisubharmonic function $\psi_{\sigma,m}$ on $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ by

$$\psi_{\sigma,m}(x^1, x^2, \dots, x^n, z) = \log\left(\prod_{j=1}^{n} |x^j|^{2\langle m - m_\sigma, v_j \rangle}\right) + \varphi_{L_0 \otimes \mathcal{L}^m}(z),$$

where U is a sufficiently small open set in V and we denote by $\varphi_{L_0\otimes\mathcal{L}^m}$ the function $\varphi_{L_0}+\sum_{j=1}^{n}\langle m, v_j\rangle\varphi_{\mathcal{L}^{v_j}}$. For the definition of φ_{L_0} and $\varphi_{\mathcal{L}^{v_j}}$, see Proposition 2.5. And here, we formally regard 0^0 as 1.

Remark 4.2. In Definition 4.1, the first term of the defining equation of $\psi_{\sigma,m}$ is clearly plurisubharmonic. According to Proposition 2.5, the second term is also turned out to be plurisubharmonic. From these arguments, we can conclude that $\psi_{\sigma,m}$ is a plurisubharmonic function, indeed.

Remark 4.3. The functions $\{e^{-\psi_{\sigma,m}}\}_{\sigma\in\Sigma_{\max}}$ glue together to give a singular hermitian metric of L. Here, we explain this fact when m is a rational point of $M_{\mathbb{R}}$ for simplicity.

Let ν be a natural number such that $\nu m \in M$. By Observation 3.8, $\nu\psi_{\sigma,m}$ can be rewritten as

$$\nu\psi_{\sigma,m} = \log|\chi^{\nu m}|^2 + \nu\varphi_{L_0\otimes\mathcal{L}^m}.$$

Since $\chi^{\nu m}$ can be regarded as a meromorphic section of the line bundle $\mathcal{O}(D_{\nu h})\otimes\pi^*\mathcal{L}^{-\nu m}$, the first term of the right hand side of the above equation is turned out to be a local weight of a singular hermitian metric which is defined globally on $\mathcal{O}(D_{\nu h})\otimes\pi^*\mathcal{L}^{-\nu m}$. Since the second term is also a local weight of the hermitian metric globally defined on $\pi^*(L_0^{\nu}\otimes\mathcal{L}^{\nu m})$, the sum $\nu\psi_{\sigma,m}$ is a local weight of a singular hermitian metric globally defined on $\nu L = \pi^*L_0^{\nu}\otimes\mathcal{O}(D_{\nu h})$.

This explanation also makes sense in the general case, by considering formally with $\mathbb{R}$-line bundles.

Definition 4.4. We define the plurisubharmonic function ψ_σ on $\mathbb{T}_N(\sigma,\mathcal{L})|_U$ by

$$\psi_\sigma(x^1,x^2,\dots,x^n,z) = \max_{m\in\square_{\mathrm{Nef}}(L_0,h)}\psi_{\sigma,m}(x^1,x^2,\dots,x^n,z)$$

for a sufficiently small open set U of V and $\sigma\in\Sigma_{\max}$.

Remark 4.5. Since each $\psi_{\sigma,m}$ is plurisubharmonic, it is clear that the upper envelope

$$(x^1,x^2,\dots,x^n,z)\longmapsto \limsup_{(\xi^1,\xi^2,\dots,\xi^n,\zeta)\to(x^1,x^2,\dots,x^n,z)}\psi_\sigma(\xi^1,\xi^2,\dots,\xi^n,\zeta)$$

of ψ_σ is a plurisubharmonic function. But here, the function

$$((x^1,x^2,\dots,x^n,z),m)\longmapsto e^{\psi_{\sigma,m}(x^1,x^2,\dots,x^n,z)} = \left(\prod_{j=1}^{n}|x^j|^{2\langle m-m_\sigma,v_j\rangle}\right)\cdot e^{\varphi_{L_0\otimes\mathcal{L}^m}(z)}$$

is a continuous function on $\mathbb{T}_N(\sigma,\mathcal{L})|_U\times\square_{\mathrm{Nef}}(L_0,h)$, and $\square_{\mathrm{Nef}}(L_0,h)$ is compact, according to Lemma 3.6. This yields that the function

$$((x^1,x^2,\dots,x^n,z),m)\longmapsto e^{\psi_\sigma(x^1,x^2,\dots,x^n,z)} = \max_{m\in\square_{\mathrm{Nef}}(L_0,h)} e^{\psi_{\sigma,m}(x^1,x^2,\dots,x^n,z)},$$

is also continuous. Therefore, ψ_σ itself is also a plurisubharmonic function.

Remark 4.6. Remark 4.3 yields that $\{e^{-\psi_\sigma}\}_{\sigma\in\Sigma_{\max}}$ glue together to give a singular hermitian metric of L whose curvature current is semi-positive.

Theorem 4.7. *Assume that L is a big line bundle, then the singular hermitian metric $e^{-\psi_\sigma}$ of L is a minimal singular metric.*

From now on, we will prepare for the proof of Theorem 4.7. Let $\sigma \in \Sigma$ be a n-dimensional cone, $v_1, v_2, \dots, v_n$ be N-minimal generators of σ, and $(x^1, x^2, \dots, x^n, z)$ be the canonical coordinate of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \dots, v_n$, where U is a sufficiently small open set in V. We use these notations throughout this section.

Lemma 4.8. *Let us fix a point $(x_0^1, x_0^2, \dots, x_0^n, z_0) \in \mathbb{T}_N(\sigma, \mathcal{L})|_U$, and denote by I the set $\{j \in \{1, 2, \dots, n\} \mid x_0^j = 0\}$. Then, there exist constants C_1 and C_2 such that*

$$\max_{m \in \square_{\mathrm{Nef}}(L_0, h)} \log \prod_{j \in I} |x^j|^{2\langle m - m_\sigma, v_j\rangle} + C_1 \leq \psi_\sigma \leq \max_{m \in \square_{\mathrm{Nef}}(L_0, h)} \log \prod_{j \in I} |x^j|^{2\langle m - m_\sigma, v_j\rangle} + C_2$$

holds on $\prod_{j\in I}^n \{|x^j| \leq 1\} \times \prod_{j \notin I} \{|x^j - x_0^j| \leq \delta_j\} \times \overline{U}$, where $\{\delta_j\}_{j \notin I}$ is a system of sufficiently small positive numbers such that $0 \notin \{|x^j - x_0^j| \leq \delta_j\}$ for all $j \notin I$, and m_σ is the element of the Cartier data of D_h which is associated to σ.

Proof. The function

$$\big(m, (x^j)_{j \notin I}, z\big) \longmapsto \log \prod_{j \notin I} |x^j|^{2\langle m - m_\sigma, v_j\rangle} + \varphi_{L_0 \otimes \mathcal{L}^m}(z)$$

defined on $\square_{\mathrm{Nef}}(L_0, h) \times \prod_{j \notin I} \{|x^j - x_0^j| \leq \delta_j\} \times \overline{U}$ is continuous. According to Lemma 3.6, $\square_{\mathrm{Nef}}(L_0, h) \times \prod_{j \notin I} \{|x^j - x_0^j| \leq \delta_j\} \times \overline{U}$ is compact, which yields that this function has both the maximum value and the minimum value, which we denote by C_1 and C_2 respectively. Therefore, the inequality

$$\log \prod_{j \in I} |x^j|^{2\langle m - m_\sigma, v_j\rangle} + C_1 \leq \psi_{\sigma, m} \leq \log \prod_{j \in I} |x^j|^{2\langle m - m_\sigma, v_j\rangle} + C_2$$

follows. We obtain the inequality of the statement by considering the maximum value of each term of the above inequality with respect to $m \in \square_{\mathrm{Nef}}(L_0, h)$. $\square$

As we have set that L is big, thus is especially pseudo-effective, there must be a minimal singular metric of L. We fix one of these and denote it by $h_{\min}$.

Lemma 4.9. *Let σ be an element of $\Sigma_{\max}$, and we denote the weight function of $h_{\min}$ around $\mathbb{T}_N(\sigma, \mathcal{L})|_{\overline{U}}$ with respect to the local trivialization of L as in Observation 3.8 by $\varphi_{\min,\sigma}$. Then, there exists a constant C_σ such that*

$$\varphi_{\min,\sigma} \leq \psi_\sigma + C_\sigma$$

holds on the set $K_\sigma = \{(x^1, x^2, \dots, x^n, z) \in \mathbb{T}_N(\sigma, \mathcal{L})|_{\overline{U}} \mid \forall j \in \{1, 2, \dots, n\}, |x^j| \leq 1\}$.

Proof. Let us denote by m_σ the element of the Cartier data of D_h associated to σ. Applying Lemma 4.8 with $I = \{1, 2, \dots, n\}$, it follows that there exists a constant C such that

$$\max_{m \in \square_{\mathrm{Nef}}(L_0, h)} \log \prod_{j=1}^n |x^j|^{2\langle m - m_\sigma, v_j\rangle} \leq \psi_\sigma + C$$

holds on K_σ.

Thus here, we compare $\varphi_{\min,\sigma}$ with $\max_{m\in\Box_{\mathrm{Nef}}(L_0,h)}\log\prod_{j=1}^n|x^j|^{2\langle m-m_\sigma,v_j\rangle}$.

We choose an infinite subsequence $\{\nu\}\subset\mathbb{N}$ and a finite subset $\{f_j^{(\nu)}\}_{1\le j\le N_\nu}$ of $H^0(X,\nu L)$ for each ν satisfying the following condition; The function

$$\varphi_\nu=\frac{1}{\nu}\log\sum_{j=1}^{N_\nu}|f_j^{(\nu)}|^2$$

converges pointwise to $\varphi_{\min,\sigma}$ on X except a subset of measure 0 as $\nu\to\infty$, and the maximum value M_{φ_ν} of φ_ν on K_σ also converges to $M_{\varphi_{\min,\sigma}}=\max_{K_\sigma}\varphi_{\min,\sigma}$ as $\nu\to\infty$. The existence of these functions can be soon showed by applying Theorem 2.2 regarding φ in the theorem as $(1-\frac{1}{k})\varphi_{\min}+\frac{1}{k}\varphi_+$ for each natural number k, where φ_+ is the local weight of a singular hermitian metric h_+ on L which satisfies $\sqrt{-1}\Theta_{h_+}\ge\varepsilon\omega$ for some positive number ε and a Kähler metric ω on X, whose existence was stated in Proposition 3.1.

Then, according to the next Lemma 4.10, the inequality

$$\varphi_\nu\le\max_{m\in\Box_{\mathrm{Nef}}(L_0,h)}\log\prod_{j=1}^n|x^j|^{2\langle m-m_\sigma,v_j\rangle}+M_{\varphi_\nu}$$

holds on K_σ. Considering this inequality as $\nu\to\infty$, we obtain

$$\varphi_{\min,\sigma}\le\max_{m\in\Box_{\mathrm{Nef}}(L_0,h)}\log\prod_{j=1}^n|x^j|^{2\langle m-m_\sigma,v_j\rangle}+M_{\varphi_{\min,\sigma}}$$

on K_σ except the subset of measure 0. But, since the both hand sides are plurisubharmonic, this inequality holds on whole K_σ.

According to the above argument, we obtain the inequality

$$\varphi_{\min,\sigma}\le\psi_\sigma+C+M_{\varphi_{\min,\sigma}}$$

on K_σ, which proves the lemma. □

Lemma 4.10. *Here, we use notations which appeared in the proof of Lemma 4.9. The inequality*

$$\varphi_\nu\le\max_{m\in\Box_{\mathrm{Nef}}(L_0,h)}\log\prod_{j=1}^n|x^j|^{2\langle m-m_\sigma,v_j\rangle}+M_{\varphi_\nu}$$

holds on K_σ.

Proof. Let us set $P(\varphi_\nu)_\sigma=\frac{1}{\nu}P(f_1^{(\nu)},f_2^{(\nu)},\dots,f_{N_\nu}^{(\nu)})_{(0,0,\dots,0,z_0)}$. According to Proposition 3.7 and Observation 3.12, $\nu P(\varphi_\nu)_\sigma$ is a subset of $S(L_0^\nu,\nu h)_\sigma$. Since $\Box_{\mathrm{Nef}}(L_0^\nu,\nu h)=\nu\Box_{\mathrm{Nef}}(L_0,h)$ holds, it tuns out that $S(L_0^\nu,\nu h)_\sigma=\nu S(L_0,h)_\sigma$, thus we obtain

$$P(\varphi_\nu)_\sigma\subset S(L_0,h)_\sigma.$$

Therefore, according to Remark 3.21, it is sufficient to show the inequality

$$\varphi_\nu \leq \max_{m \in P(\varphi_\nu)_\sigma} \log \prod_{j=1}^{n} |x^j|^{2\langle m, v_j \rangle} + M_{\varphi_\nu}$$

on K_σ.

According to Remark 3.24, there exists a finite subset A of $P(\varphi_\nu)$ whose elements are rational and which satisfies $P(\varphi_\nu) = \overline{\overline{A}}$. For this set A, we fix such a subdivision $\tilde{\Sigma}$ of Σ as in Lemma 3.25. In the following, we use notations we used in Section 4.2. According to Corollary 3.18, it is sufficient to show that

$$\mu^* \varphi_\nu \leq \mu^* \left(\max_{m \in P(\varphi_\nu)_\sigma} \log \prod_{j=1}^{n} |x^j|^{2\langle m, v_j \rangle} \right) + M_{\varphi_\nu}$$

on $K_{\tilde{\sigma}} = \{(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n, z) \in \mathbb{T}_N(\tilde{\sigma}, \mathcal{L})|_{\overline{U}} \mid \forall j \in \{1, 2, \dots, n\}, |\tilde{x}^j| \leq 1\}$ for each $\tilde{\sigma} \in (\tilde{\Sigma}_\sigma)_{\max}$.

Since

$$\begin{aligned} \log \prod_{j=1}^{n} |\mu^* x^j|^{2\langle m, v_j \rangle} &= \log \prod_{j=1}^{n} \prod_{k=1}^{n} |\tilde{x}^k|^{2\langle m, v_j \rangle \langle v^j, \tilde{v}_k \rangle} \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} \langle m, v_j \rangle \langle v^j, \tilde{v}_k \rangle \log |\tilde{x}^k|^2 \\ &= \sum_{k=1}^{n} \langle m, \tilde{v}_k \rangle \log |\tilde{x}^k|^2 \end{aligned}$$

holds, we obtain

$$\mu^* \left(\max_{m \in P(\varphi_\nu)_\sigma} \log \prod_{j=1}^{n} |x^j|^{2\langle m, v_j \rangle} \right) = \max_{m \in P(\varphi_\nu)_\sigma} \sum_{j=1}^{n} \langle m, \tilde{v}_j \rangle \log |\tilde{x}^j|^2.$$

As $\log |\tilde{x}^j|^2 \leq 0$ holds for all j on $K_{\tilde{\sigma}}$, the equation we desire can be rewritten as

$$\mu^* \varphi_\nu \leq \log \prod_{j=1}^{n} |\tilde{x}^j|^{2\langle m_0, \tilde{v}_j \rangle} + M_{\varphi_\nu},$$

where $m_0 \in P(\varphi_\nu)_\sigma$ is such an element as in Lemma 3.25.

Let us set $P(\varphi_\nu)_{\tilde{\sigma}} = \frac{1}{\nu} P(\mu^* f_1^{(\nu)}, \mu^* f_2^{(\nu)}, \dots, \mu^* f_{N_\nu}^{(\nu)})_{(0,0,\dots,0,z_0)}$. According to Remark 3.23, and since both $P(\varphi_\nu)_{\tilde{\sigma}}$ and $P(\varphi_\nu)_\sigma$ are generated by the same set, it turns out that $\mu^* f_j^{(\nu)}$ can be divided by the function $\prod_{k=1}^{n} (x^k)^{\langle \nu m_0, \tilde{v}_k \rangle}$ for all $j \in \{1, 2, \dots, N_\nu\}$. Denoting the

quotient by $g_j^{(\nu)}$, the function $\mu^*\varphi_\nu - \log\prod_{j\in I}|\tilde{x}^j|^{2\langle m_0,\tilde{v}_j\rangle}$ can be rewritten as

$$\mu^*\varphi_\nu - \log\prod_{j=1}^{n}|\tilde{x}^j|^{2\langle m_0,\tilde{v}_j\rangle} = \frac{1}{\nu}\log\sum_{j=1}^{N_\nu}|g_j^{(\nu)}|^2.$$

Thus, this function is a plurisubharmonic function on $K_{\tilde{\sigma}}$, and it has the maximum value on $K_{\tilde{\sigma}}$, which we denote by $M_{\varphi_\nu,\tilde{\sigma}}$. Then, since

$$\mu^*\varphi_\nu \leq \log\prod_{j=1}^{n}|\tilde{x}^j|^{2\langle m_0,\tilde{v}_j\rangle} + M_{\varphi_\nu,\tilde{\sigma}}$$

holds on $K_{\tilde{\sigma}}$. Therefore, it remains to prove that $M_{\varphi_\nu,\tilde{\sigma}} \leq M_{\varphi_\nu}$.

Let the plurisubharmonic function $\mu^*\varphi_\nu - \log\prod_{j\in I}|\tilde{x}^j|^{2\langle m_0,\tilde{v}_j\rangle}$ hae the maximum value at the point $(\tilde{x}_0^1, \tilde{x}_0^2, \dots, \tilde{x}_0^n, z_0) \in K_{\tilde{\sigma}}$. Then, it turns out that we may assume $|\tilde{x}_0^j| = 1$ for all j after we change the point $(\tilde{x}_0^1, \tilde{x}_0^2, \dots, \tilde{x}_0^n, z_0) \in K_{\tilde{\sigma}}$ if necessary. It is because, when $|\tilde{x}_0^1| < 1$ for example, let us consider the function

$$\tilde{x}^1 \longmapsto \mu^*\varphi_\nu(\tilde{x}^1, \tilde{x}_0^2, \tilde{x}_0^3, \dots, \tilde{x}_0^n, z_0) - \log\left(|\tilde{x}^1|^{2\langle m_0,\tilde{v}_1\rangle}\cdot\prod_{j=2}^{n}|\tilde{x}_0^j|^{2\langle m_0,\tilde{v}_j\rangle}\right).$$

Since this is a plurisubharmonic function on $\{|\tilde{x}^1| < 1\}$, and according to the principle of the maximum, the value of the function above must constantly be $M_{\varphi_\nu,\tilde{\sigma}}$. Thus, in this case, we can change the point to another one whose coordinate satisfies th condition $|\tilde{x}_0^j| = 1$.

Then, we can calculate that

$$M_{\varphi_\nu,\tilde{\sigma}} = \mu^*\varphi_\nu(\tilde{x}_0^1, \tilde{x}_0^2, \dots, \tilde{x}_0^n, z_0) - \log\prod_{j=1}^{n}|\tilde{x}_0^j|^{2\langle m_0,\tilde{v}_j\rangle} = \varphi_\nu(\mu(\tilde{x}_0^1, \tilde{x}_0^2, \dots, \tilde{x}_0^n, z_0)).$$

Since $\mu(\tilde{x}_0^1, \tilde{x}_0^2, \dots, \tilde{x}_0^n, z_0) \in K_\sigma$, the value is at most M_{φ_ν}. $\square$

Proof of Proposition 4.7. Let us denote by h the singular hermitian metric defined by $\{e^{-\psi_\sigma}\}_\sigma$, and by h_∞ a smooth hermitian metric on L. Then, there exists upper semi-continuous functions $\varphi'_{\min}$ and ψ' on X such that

$$h_{\min} = h_\infty e^{-\varphi'_{\min}},\ h = h_\infty e^{-\psi'}$$

hold. Here, it is sufficient to prove that there exists a constant C such that

$$\varphi'_{\min} \leq \psi' + C$$

holds on $\pi^{-1}(\overline{U}) \subset X$.

According to Lemma 4.9, for each $\sigma \in \Sigma_{\max}$, there exists a constant C_σ such that

$$\varphi'_{\min} \leq \psi' + C_\sigma$$

holds on the set $K_\sigma = \{(x^1, x^2, \dots, x^n, z) \in \mathbb{T}_N(\Sigma, \mathcal{L})|_{\overline{U}} \mid \forall j \in \{1, 2, \dots, n\}, |x^j| \leq 1\}$. Thus, according to Lemma 3.13,

$$\varphi'_{\min} \leq \psi' + C$$

holds on $\pi^{-1}(\overline{U}) \subset X$, where $C = \max_{\sigma \in \Sigma_{\max}} C_\sigma$. □

5. Properties related to the singularities of minimal singular metrics

5.1. Kiselman numbers and Lelong numbers of minimal singular metrics and Non-nef loci. Let X be a smooth projective variety over $\mathbb{C}$ and L be a line bundle over X. According to [3] 3.6, the next proposition follows.

Proposition 5.1. *If L is big, then the non-nef locus $\mathrm{NNef}(L)$ of L can be written as*

$$\mathrm{NNef}(L) = \{x \in X \mid \nu(\varphi_{\min}, x) > 0\},$$

where $e^{-\varphi_{\min}}$ is a minimal singular metric of L.

According to this proposition, we can specify the non-nef locus of a big line bundle by calculating the Lelong number of a minimal singular metric. It can be done, actually, in our settings.

Proposition 5.2. *Let X be the total space of a toric bundle $\mathbb{T}_N(\Sigma, \mathcal{L})$ over a complex torus and $L = \pi^* L_0 \otimes \mathcal{O}(D_h)$ be a big line bundle over X, where Σ is a smooth projective fan in a n-dimensional lattice N. The Kiselman number*

$$\nu^K_{\zeta, w}(\varphi_{\min}, x_0) = \sup \left\{ t \geq 0 \,\middle|\, \varphi_{\min} \leq t \log \sum_{j=1}^{n+d} |\zeta^j|^{2w_j} + \mathcal{O}(1) \text{ around } x_0 \right\}$$

associated to the coordinate $\zeta = (\zeta^1, \zeta^2, \dots, \zeta^{n+d}) = (x^1, x^2, \dots, x^n, z^1, z^2, \dots, z^d)$ and $w = (w_j) \in \bigoplus_{j \in I} \mathbb{R}_{\geq 0}$ of a minimal singular metric $e^{-\varphi_{\min}}$ at a point $x_0 = (x_0^1, x_0^2, \dots, x_0^n, z_0) \in \mathbb{T}_N(\sigma, \mathcal{L})$ (see [1] Section 5.2 for details) can be calculated that

$$\nu^K_{\zeta, w}(\varphi_{\min}, x_0) = \min_{m \in S(L_0, h)_\sigma} \left\langle m, \sum_{j \in I} \frac{v_j}{w_j} \right\rangle,$$

where we denote by I the set $\{j \mid x_0^j = 0\}$ and by $(x^1, x^2, \dots, x^n, z^1, z^2, \dots, z^d)$ the canonical coordinate of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to N-minimal generators $v_1, v_2, \dots, v_n$ of σ. Especially, the Lelong number at x_0 can be calculated that

$$\nu(\varphi_{\min}, x_0) = \min_{m \in S(L_0, h)_\sigma} \sum_{j \in I} \langle m, v_j \rangle.$$

Here, we are using notations in the previous section.

Proof. Applying Theorem 4.7 with Lemma 4.8 and Remark 3.21, it is obvious that

$$\nu^K_{\zeta, w}(\varphi_{\min}, x_0) \quad = \quad \nu^K_{\zeta, w}\left(\log \max_{m \in S(L_0, h)_\sigma} \prod_{j \in I} |x^j|^{2\langle m, v_j \rangle}, x_0 \right).$$

Thus, here we consider the condition

$$\log \max_{m\in S(L_0,h)_\sigma} \prod_{j\in I} |x^j|^{2\langle m,v_j\rangle} \le t\log\sum_{j=1}^{n+d}|\zeta^j|^{2w_j} + \mathcal{O}(1)$$

around x_0 for a nonnegative number t.

Let us set $l=\sqrt{\sum_{j=1}^{n+d}|\zeta^j|^{2w_j}}$ and define a point $(s_j)_{j\in I}$ of the $(n+d-1)$-dimensional sphere S^{n+d-1} by $|\zeta^j| = l\cdot s_j$ for all j and $(\zeta^1,\zeta^2,\cdots,\zeta^n)\neq x_0$. Then the above condition can be rewritten as

$$\log \max_{m\in S(L_0,h)_\sigma} \left(l^{2\left\langle m,\sum_{j\in I}\frac{v_j}{w_j}\right\rangle} \prod_{j\in I} s_j^{2\left\langle m,\frac{v_j}{w_j}\right\rangle}\right) \le t\log l^2 + \mathcal{O}(1).$$

Here, let $m_0\in S(L_0,h)_\sigma$ be an element which realize the minimum

$$\min_{m\in S(L_0,h)_\sigma}\sum_{j\in I}\left\langle m,\frac{v_j}{w_j}\right\rangle.$$

By using this m_0, the left hand side of the above inequality can be rewritten as

$$\left\langle m_0,\sum_{j\in I}\frac{v_j}{w_j}\right\rangle \log l^2 + \log\max_{m\in S(L_0,h)_\sigma}\left(l^{2\left\langle m-m_0,\sum_{j\in I}\frac{v_j}{w_j}\right\rangle}\prod_{j\in I} s_j^{2\left\langle m,\frac{v_j}{w_j}\right\rangle}\right)$$

Since the obvious relation

$$\log\prod_{j\in I} s_j^{2\left\langle m_0,\frac{v_j}{w_j}\right\rangle} \le \log\max_{m\in S(L_0,h)_\sigma}\left(l^{2\left\langle m-m_0,\sum_{j\in I}\frac{v_j}{w_j}\right\rangle}\prod_{j\in I} s_j^{2\left\langle m,\frac{v_j}{w_j}\right\rangle}\right)$$

holds, the condition can be rewritten as

$$\left\langle m_0,\sum_{j\in I}\frac{v_j}{w_j}\right\rangle \log l^2 \le t\log l^2 + \mathcal{O}(1),$$

which proves the proposition. □

Corollary 5.3. *Let X, L be as that of the previous proposition. The following conditions related to the behavior of the minimal singular metric $e^{-\varphi_{\min}}$, $\mathbb{T}_N(\sigma,\mathcal{L})$ of L over X at a point x_0 are equivalent.*

(1) $\varphi_{\min}(x_0)(=\psi_\sigma(x_0)) = -\infty$.
(2) ψ_σ *is not continuous at* x_0.
(3) $\nu(\varphi_{\min},x_0)(=\nu(\psi_\sigma,x_0))>0$.

especially,

$$\varphi_{\min}^{-1}(-\infty) = \mathrm{Pole}(\varphi_{\min})$$

holds, where we denote by $\mathrm{Pole}(\varphi_{\min})$ *the set* $\{x\in X \mid \nu(\varphi_{\min},x)>0\}$.

The next proposition is also obtained easily by Theorem 4.7.

Proposition 5.4. *Let X, L be as that of Proposition 5.2. Then,* $\mathrm{Pole}(\varphi_{\min})$ *is a Zariski-closed set.*

Proof. Here, we use notations appeared in the proof of the Theorem 4.7. Then, the set $\mathrm{Pole}(\psi_\sigma)$ can be written as

$$\mathrm{Pole}(\varphi_{\min}) = \bigcap_{j=1}^{l} \bigcup_{k\in I_j} \{x^k = 0\}$$

on $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ for some $I_1, I_2, \dots, I_l \subset \{1, 2, \dots, n\}$, and this proves the proposition. $\square$

According to these argument, we obtain the following corollary.

Corollary 5.5. *Let X be the total space of a toric bundle $\mathbb{T}_N(\Sigma, \mathcal{L})$ over a complex torus and $L = \pi^* L_0 \otimes \mathcal{O}(D_h)$ be a big line bundle over X, where Σ is a smooth projective fan. Then, the set* $\mathrm{NNef}(L)$ *is Zariski-closed subset of X.*

5.2. **Multiplier ideal sheaves.** Let Σ be a smooth projective fan of a n-dimensional lattice N. Let us fix N-minimal generators $v_1, v_2, \dots, v_n$ of $\sigma \in \Sigma_{\max}$, and let $(x^1, x^2, \dots, x^n, z)$ be the canonical coordinate of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$ associated to $v_1, v_2, \dots, v_n$, where U is a sufficiently small open set in V. In this section, we consider the condition

$$f \in \mathcal{J}(h_{\min}^t)_{(x_0^1, x_0^2, \dots, x_0^n, z_0)},$$

where $(x_0^1, x_0^2, \dots, x_0^n, z_0)$ is a point of $\mathbb{T}_N(\sigma, \mathcal{L})|_U$, f is an element of $\mathcal{O}_{X,(x_0^1, x_0^2, \dots, x_0^n, z)} \setminus \{0\}$, t is a positive real number, and $h_{\min}$ is a minimal singular metric of L. In the following, we also denote by $\mathcal{J}(t\varphi_{\min})$ the multiplier ideal sheaf $\mathcal{J}(h_{\min}^t)$ by using the local weight function $\varphi_{\min}$ of the singular hermitian metric $h_{\min}$.

Let us set $I = \{j \in \{1, 2, \dots, n\} \mid x_0^j = 0\}$. For this set I, let us denote the expansion appeared in Definition 3.22 by

$$\begin{aligned} f(x^1, x^2, \dots, x^n, z) &= \sum_{(a_j)_j \in \mathbb{Z}_{\geq 0}^I} \prod_{j\in I} (x^j)^{a_j} A_{\sum_j a_j v^j}(x^{I^c}, z) \\ &= \sum_{m \in \mathrm{Pr}^I(\sigma^\vee \cap M)} \prod_{j\in I} (x^j)^{\langle m, v_j\rangle} A_m(x^{I^c}, z), \end{aligned}$$

where the map Pr^I stands for the projection from $M_\mathbb{R}$ to $\mathrm{Span}_\mathbb{R}\{v^j\}_{j\in I}$. As the dual version of this map, we denote the projection from $N_\mathbb{R}$ to $\mathrm{Span}_\mathbb{R}\{v_j\}_{j\in I}$ by Pr_I in the following. Let $A \subset P(f)_(x_0^1, x_0^2, \dots, x_0^n, z_0)$ be a set of lattice points such that

$$P(f)_{(x_0^1, x_0^2, \dots, x_0^n, z_0)} = \overline{\overline{A}}$$

holds.

Corollary 5.6. *In the above setting, the following are equivalent.*

(1) $f \in \mathcal{J}(t\varphi_{\min})_{(x_0^1, x_0^2, \ldots, x_0^n, z_0)}$.

(2) $\min_{m \in tS(L_0,h)_\sigma} \langle m, w \rangle < \langle m_0 + \sum_{j \in I} v^j, w \rangle$ *for all* $m_0 \in A$ *and* $w \in \mathrm{Pr}_I(\sigma) \setminus \{0\}$.

Corollary 5.6 soon follows from Theorem 4.7 and the next theorem, which is the special version of the result of Guenancia [10] referring to the way to compute the multiplier ideal sheaves associated to "toric plurisubharmonic functions". More preciously, we can obtain Corollary 5.6 by applying Theorem 5.7 with regarding $(x^j)_{j=1}^n$ in the theorem as $(x^j)_{j \in I}$ in the corollary, and $(z^j)_{j=1}^d$ in the theorem as $((x^j)_{j \notin I}, (z^j)_{j=1}^d)$ in the corollary, and repeating the same argument as in proof of Lemma 4.8. This theorem can be regarded as a generalization of the famous Howald's result ([11] Theorem 11) in algebraic settings.

Theorem 5.7. *(The special version of* [10] *Theorem A) Let* S *be a compact subset of* $\bigoplus_{j=1}^n \mathbb{R}_{\geq 0} v^j$ *and let us consider the plurisubharmonic function*

$$\psi(x, z) = \psi(x^1, x^2, \ldots, x^n, z^1, z^2, \ldots, z^d) = \log \max_{m \in S} \prod_{j=1}^n |x^j|^{\langle m, v_j \rangle}$$

defined on the neighborhood of $0 \in \mathbb{C}^{n+d}$*, where* $v_1, v_2, \ldots, v_n$ *is the dual basis of* $(v^j)_j$*. Let* f *be an element of the stalk* $\mathcal{O}_{\mathbb{C}^{n+d},0}$ *and let us denote the Taylor expansion of* f *for the variables* $x = (x^j)_j$ *be*

$$f(x, z) = \sum_{m \in \bigoplus_{j=1}^n \mathbb{Z}_{\geq 0} v^j} \prod_{j=1}^n (x^j)^{\langle m, v_j \rangle} A_m(z),$$

where each A_m *is an element of the stalk* $\mathcal{O}_{\mathbb{C}^d,0}$*. Then, the following are equivalent.*

(1) $f \in \mathcal{J}(\psi)_0$.

(2) $\min_{m \in S} \langle m, w \rangle < \langle m_0 + \sum_{j=1}^n v^j, w \rangle$ *for all* $m_0 \in S$ *and* $w \in \bigoplus_{j=1}^n \mathbb{R}_{\geq 0} v_j \setminus \{0\}$.

According to Corollary 5.6, [5] 1.10, 1.11, and [13] 11.2.12 (ii), we obtain the next corollary.

Corollary 5.8. *Let* X *be the total space of a smooth projective toric bundle over a complex torus,* D *be a big divisor on* X*, and* $e^{-\varphi_{\min}}$ *be a minimal singular metric on the line bundle* $\mathcal{O}(D)$.

(1) If $f \in \mathcal{J}(t\varphi_{\min})_{x_0}$ *at the point* x_0*, then* $f \in \mathcal{J}((1+\varepsilon)t\varphi_{\min})_{x_0}$ *holds for sufficiently small positive number* ε *and any positive real number* t*. Especially, since the sheaf* $\mathcal{J}(t\varphi_{\min})$ *is coherent, it follows that*

$$\mathcal{J}(t\varphi_{\min}) = \mathcal{J}_+(t\varphi_{\min}).$$

(2) Let P be a nef big divisor on X, then

$$H^j(X, \mathcal{O}(K_X + P + L) \otimes \mathcal{J}(\varphi_{\min})) = 0$$

holds for all $j > 0$.

The next conjecture is by Demailly and Kollár.

Conjecture 5.9. *(A weak form of the openness conjecture [7]) Let X be a complex manifold, x be a point of X, and φ be a plurisubharmonic function defined on some neighborhood of x. Then,*

$$\{t > 0 \mid e^{-t\varphi} \text{ is integrable around the point } x\} = (0, c_x(\varphi))$$

holds.

Corollary 5.10. *Conjecture 5.9 is affirmative in the setting that X is the total space of a smooth projective toric bundle over a complex torus and φ is the local weight of a minimal singular metric of a big line bundle over X.*

6. Some examples

In this section, we will introduce three examples for X and L in the previous sections. We construct them as $\mathbb{P}^2$-bundles over abelian surfaces, by following [14] CHAPTER IV §2.6 basically. In this section, we use notations we introduced in Example 2.10.

As a preparation, we first recall a useful lemma to see L is big.

Lemma 6.1. *In the setting of Example 2.10, L is big if and only if there exists a triple (a, b, c) of nonnegative integers such that $L_0^a \otimes L_1^b \otimes L_2^c$ is ample line bundle over V.*

This lemma can be easily shown by applying the result known by Cutkosky ([12], Lemma 2.3.2) and the fact that the ample cones of complex tori coincide with these big cones.

Let E be a sufficiently general smooth elliptic curve and o be a point of E. For example, you can choose $\mathbb{C}/(\mathbb{Z} + (\pi + \sqrt{-1})\mathbb{Z})$ for E. Let

$$V = E \times E.$$

It is known that the rank of the Neron-Severi group $\mathrm{NS}(V)$ of V is three and this group is generated by the following three classes ([12] Chapter 1.5.B).

- $f_1 = c_1(\mathcal{O}_V(F_1))$, where F_1 stands for the prime divisor $\{o\} \times E \subset V$.
- $f_2 = c_1(\mathcal{O}_V(F_2))$, where F_2 stands for the prime divisor $E \times \{o\} \subset V$.
- $\delta = c_1(\mathcal{O}_V(\Delta))$, where Δ stands for the prime divisor $\{(x, y) \in E \times E \mid x = y\}$.

By using these three classes, the nef cone $\mathrm{Nef}(V)$ of V can be written as

$$\mathrm{Nef}(V) = \{af_1 + bf_2 + c\delta \mid a, b, c \in \mathbb{R},\ ab + bc + ca \geq 0,\ a + b + c \geq 0\}.$$

In order to obtain more useful expression of $\mathrm{Nef}(V)$, let us define the other basis of $\mathrm{NS}(V) \otimes \mathbb{R}$ by

$$l_1 = \frac{1}{6}(f_1 + f_2 - 2\delta),\ l_2 = \frac{1}{6}(-\sqrt{3}f_1 + \sqrt{3}f_2),\ \text{and } l_3 = \frac{1}{6}(f_1 + f_2 + \delta).$$

By using these classes, $\mathrm{Nef}(V)$ can be written as

$$\mathrm{Nef}(V) = \{al_1 + bl_2 + cl_3 \mid c^2 \geq a^2 + b^2,\ c \geq 0\}.$$

This expression of $\mathrm{Nef}(V)$ makes it easy to judge the nef-ness of line bundles.

Example 6.2. The first example is an example which admits a Zariski-decomposition after appropriate proper modifications. Let us fix two positive natural numbers $u < v$ and set

$$\begin{aligned} L_0 &= \mathcal{O}(-uF_1 - uF_2 - u\Delta), \\ L_1 &= \mathcal{O}((u+v)F_1 + (u+v)F_2 + (-2u+v)\Delta), \end{aligned}$$

and

$$L_2 = \mathcal{O}((-u+v)F_1 + (-u+v)F_2 + (2u+v)\Delta).$$

Then, we can easily calculate that

$$c_1(L_0) = -6ul_3,\ c_1(L_1) = 6(ul_1 + vl_3),\ \text{and}\ c_1(L_2) = 6(-ul_1 + vl_3).$$

These expressions makes it clear that the line bundle $L_1 \otimes L_2$ is ample and, according to Lemma 6.1, that L is a big line bundle in this case.

The set $\square_{\mathrm{Nef}}(L_0, h)$ in this setting is rational polyhedral. More preciously, $\square_{\mathrm{Nef}}(L_0, h)$ is the convex closure of the five points

$$e^1,\ e^2,\ \frac{u}{v}e^2,\ \frac{u}{2(u+v)}e^1 + \frac{u}{2(u+v)}e^2,\ \frac{u}{v}e^1$$

in $M_{\mathbb{R}}$. So, by applying Theorem 4.7, it soon turns out that the weights of a minimal singular metric ψ_{σ_j}'s satisfies

$$\psi_{\sigma_j} \sim_{\mathrm{sing}} 1$$

at any points of X except for the locus $\mathbb{P}(L_0)$, and

$$\begin{aligned} \psi_{\sigma_1}(x^1, x^2, z) &\sim_{\mathrm{sing}} \frac{u}{2v(u+v)} \log\max\{|x^1|^{2(2u+2v)}, |x^2|^{2(2u+2v)}, |x^1|^{2v}|x^2|^{2v}\} \\ &\sim_{\mathrm{sing}} \frac{u}{2v(u+v)} \log\left(|x^1|^{2(2u+2v)} + |x^2|^{2(2u+2v)} + |x^1|^{2v}|x^2|^{2v}\right) \end{aligned}$$

at a point $(0, 0, z_0) \in \mathbb{P}(L_0)$. Therefore, it follows that the non-nef locus $\mathrm{NNef}(L)$ is a Zariski-closed subset $\mathbb{P}(L_0)$ of X.

According to [14] 2.5, the fact that $\square_{\mathrm{Nef}}(L_0, h)$ is a rational polyhedral yields that L admits a Zariski-decomposition after appropriate proper modifications. Especially, when u and v can be written as

$$u = 1,\ v = 2n - 2$$

for some integer $n > 1$, (X, L) is an example which admits a Zariski-decomposition just after the n-time blow-up centered at the non-nef locus of the pull-back of L. It can be also checked out by using the above expression of the minimal singular metric of L.

According to the above expression of $\square_{\mathrm{Nef}}(L_0, h)$, the result of Corollary 5.6 can be rewritten as following. First, it is clear that $\mathcal{J}(h^t_{\mathrm{min}})$ is trivial at any point in $X \setminus \mathbb{P}(L_0)$.

Next, for a point $x_0 \in \mathbb{P}(L_0)$, the stalk of $\mathcal{J}(h_{\min})_{x_0}$ of the multiplier ideal sheaf at x_0 is the ideal of $\mathcal{O}_{X,x_0}$ which is generated by the system of the polynomials

$$\{(x^1)^p(x^2)^q \mid (p+1,q+1) \in \mathrm{Int}(S_t) \cap \mathbb{Z}^2\},$$

where we denote by $\mathrm{Int}(S_t)$ the interior of the set

$$S_t = \{(\langle tm, e_1\rangle, \langle tm, e_2\rangle) \in \mathbb{R}^2 \mid m \in S(L_0,h)_{\sigma_1}\}.$$

For the detail shape of S_t, see Figure 2 below.

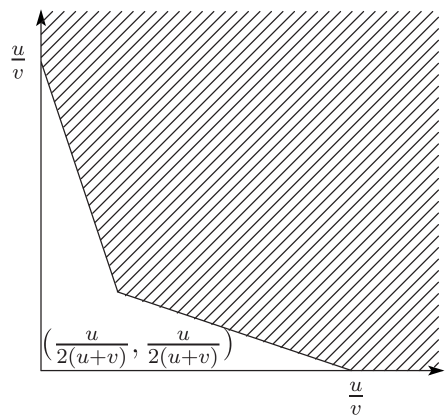

FIGURE 2. The shaded area of this figure represents the set S_1. The set S_t is the set of points $p \in \mathbb{R}^2$ which satisfies $\frac{p}{t} \in S_1$.

Finally, we introduce the result about the jumping numbers associated to the minimal singular metric. As above, we can calculate the multiplier ideal sheaves $\mathcal{J}(h_{\min}^t)$ for any positive real number t. By using this result, we can calculate that the set of the whole jumping numbers $\mathrm{Jump}(\psi_{\sigma_1}; x_0)$ at a point $x_0 \in \mathbb{P}(L_0)$ can be written as

$$\mathrm{Jump}(\psi_{\sigma_1}; x_0) = \left\{ 2p + (p+q)\frac{v}{u} \,\middle|\, p, q \in \mathbb{Z},\ 1 \leq p \leq q \right\},$$

and the singularity exponent $c_{x_0}(\psi_{\sigma_1})$, which is the least number in $\mathrm{Jump}(\psi_{\sigma_1}; x_0)$, satisfies

$$c_{x_0}(\psi_{\sigma_1}) = 2\left(1 + \frac{v}{u}\right).$$

Remark 6.3. In Example 6.2, the behavior of the multiplier ideal sheaf $\mathcal{J}(\psi_{\sigma_1})$ around a point of $\mathbb{P}(L_0)$ coincides with that of the (algebraic) multiplier ideal sheaf $\mathcal{J}(\mathfrak{a}^c)$, where $\mathfrak{a}$ is an ideal generated by $((x^1)^{2(u+v)}, (x^2)^{2(u+v)}, (x^1)^v(x^2)^v)$ and c is the rational number $\frac{u}{2v(u+v)}$.

This means that the analytic multiplier ideal sheaf $\mathcal{J}(\psi_{\sigma_1})_{x_0}$ has properties same as algebraic multiplier ideal sheaves. For example, it is known that, related to the algebraic multiplier ideal sheaf $\mathcal{J}(\mathfrak{a}^c)$, the set of the whole jumping numbers $\mathrm{Jump}(\mathfrak{a}; x_0)$ is a discrete subset of the set of rational numbers $\mathbb{Q}$, and has the property so-called "periodicity" in a sufficiently big parts of this set (see [9] 1.12 for details). Indeed, it can be easily checked that $\mathrm{Jump}(\psi_{\sigma_1}; x_0)$ is a discrete subset of $\mathbb{Q}$, and has a "period" $c^{-1} = 2v(1 + \frac{v}{u})$.

Example 6.4. As a second example, we introduce the example found out by Nakayama ([14]), which admits no Zariski-decomposition even after any proper modifications.

Let us fix a natural number $a > 1$ and set

$$\begin{aligned} L_0 &= \mathcal{O}(2F_1 - 4F_2 + 2\Delta), \\ L_1 &= \mathcal{O}((a-1)F_1 + (a-1)F_2 + (a+2)\Delta)\Delta), \end{aligned}$$

and

$$L_2 = \mathcal{O}((a+3)F_1 + (a-3)F_2 + a\Delta).$$

Then, we can easily calculate that

$$c_1(L_0) = -6(l_1 + \sqrt{3}l_2),\ c_1(L_1) = 6(-l_1 + al_3),\ \text{and } c_1(L_2) = 6(-\sqrt{3}l_2 + al_3).$$

By these expressions, it turns out that the line bundles L_1 and L_2 are ample and, according to Lemma 6.1, that L is also a big line bundle in this case.

For this setting, see Section 1.

Example 6.5. Finally, we introduce an example which can be proved that admits no Zariski-decomposition even after any proper modifications in the almost same way to the case of previous Nakayama's example, but whose minimal singular metric can be expressed more easily.

Let us set

$$\begin{aligned} L_0 &= \mathcal{O}(4F_1 + 4F_2 + \Delta), \\ L_1 &= \mathcal{O}_X, \end{aligned}$$

and

$$L_2 = \mathcal{O}(-F_1 + 9F_2 + \Delta).$$

Then, we can easily calculate that

$$c_1(L_0) = 6(l_1 + 3l_3),\ c_1(L_1) = 0,\ \text{and } c_1(L_2) = 6l_1 + 10\sqrt{3}l_2 + 18l_3.$$

By this expression, it turns out that the line bundle L_0 is ample and, from Lemma 6.1, that L is also a big line bundle in this case.

The set $\square_{\mathrm{Nef}}(L_0, h)$ in this setting is not rational, but is polyhedral. More preciously, $\square_{\mathrm{Nef}}(L_0, h)$ is the convex closure of the three points

$$0,\ e^1,\ \text{and } \frac{2\sqrt{6}}{5}e^2$$

in $M_{\mathbb{R}}$. So, applying theorem 4.7, it soon turns out that the weights of a minimal singular metric ψ_{σ_j}'s satisfies

$$\psi_{\sigma_j} \sim_{\mathrm{sing}} 1$$

at any points of X except for the locus $\mathbb{P}(L_2)$, and

$$\begin{aligned} \psi_{\sigma_3}(x^1, x^2, z) &\sim_{\mathrm{sing}} \log \max\{|x^0|^{2\alpha}, |x^1|^2\} \\ &\sim_{\mathrm{sing}} \log\left(|x^0|^{2\alpha} + |x^1|^2\right) \end{aligned}$$

at a point $(0, 0, z_0) \in \mathbb{P}(L_2)$, where we denote by α the positive irrational number $1 - \frac{2\sqrt{6}}{5}$. Therefore, it follows that the non-nef locus $\mathrm{NNef}(L)$ is a Zariski-closed subset $\mathbb{P}(L_2)$ of X.

According to the above expression of $\square_{\mathrm{Nef}}(L_0, h)$, the result of Corollary 5.6 can be rewritten as following. First, it is clear that $\mathcal{J}(h^t_{\min})$ is trivial at any point in $X \setminus \mathbb{P}(L_2)$. Next, for a point $x_0 \in \mathbb{P}(L_2)$, the stalk $\mathcal{J}(h_{\min})_{x_0}$ of the multiplier ideal sheaf at x_0 is the ideal of $\mathcal{O}_{X,x_0}$ which is generated by the system of the polynomials

$$\{(x^1)^p(x^2)^q \mid (p+1, q+1) \in \mathrm{Int}(S_t) \cap \mathbb{Z}^2\},$$

where we denote by S_t the set $\{(\langle tm, e_1\rangle, \langle tm, e_2\rangle) \in \mathbb{R}^2 \mid m \in S(L_0, h)_{\sigma_3}\}$. For the detail shape of S_t in this case, see Figure 3 below.

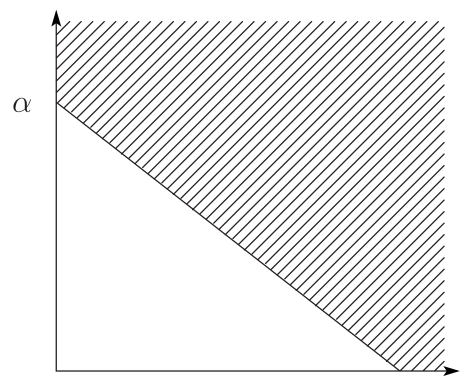

FIGURE 3. The shaded area of this figure represents the set S_1. The set S_t is the set of points $p \in \mathbb{R}^2$ which satisfies $\frac{p}{t} \in S_1$.

Let x_0 be a point in $\mathbb{P}(L_2)$. In this case, $\mathrm{Jump}(\psi_{\sigma_3}; x_0)$ can be calculated that

$$\mathrm{Jump}(\psi_{\sigma_3}; x_0) = \mathbb{Z}_{>0} \oplus \frac{1}{\alpha} \cdot \mathbb{Z}_{>0},$$

and the singularity exponent can be calculated that

$$c_{x_0}(\psi_{\sigma_1}) = 1 + \frac{1}{\alpha},$$

which is not rational, too.

Proposition 6.6. *In the above settings, L admits no Zariski-decomposition even after any proper modifications.*

Proposition 6.6 can easily be proved by making such a sequence as in Lemma 6.7.

Lemma 6.7. *(*[14] *2.11) Let us consider a countable sequence of blowing-ups*

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$$

and denote by μ_ν the blowing-up morphism $X_\nu \to X_{\nu-1}$. Assume that the center of each blowing-up μ_n is a smooth subvariety $V_n \subset X_{n-1}$ of codimension 2, and let us denote the

exceptional divisor $\mu_n^{-1}(V_n)$ *by* E_n. *We also assume that each big* $\mathbb{R}$*-divisor* D_n *of* X_n *satisfies the following three conditions;*

$$\begin{aligned} &(1) \quad \mu_n(V_{n+1}) = V_n, \\ &(2) \quad \nu(\{D_{n-1}\}, V_n) > 0, \\ &(3) \quad D_n = \mu_n^* D_{n-1} - \nu(\{D_{n-1}\}, V_n) E_n. \end{aligned}$$

Then, D_0 *admits no Zariski-decomposition even after any proper modifications.*

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